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Continuous symmetric reductions of the Adler–Bobenko–Suris equations

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Abstract
Continuously symmetric solutions of the Adler–Bobenko–Suris class of discrete integrable equations are presented. Initially defined by their invariance under the action of both of the extended three-point generalized symmetries admitted by the corresponding equations, these solutions are shown to be determined by an integrable system of partial differential equations. The connection of this system to the Nijhoff–Hone–Joshi 'generating partial differential equations' is established and an auto-Bäcklund transformation and a Lax pair for it are constructed. Applied to the *H1* and $Q1_{\delta=0}$ members of the Adler–Bobenko–Suris family, the method of continuously symmetric reductions yields explicit solutions determined by the Painlevé trancendents.

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1. Introduction

The study of integrable discrete systems has a long history going back to work in the late 1970s and early 1980s [1–4]. At this point, it is acknowledged that most of the well-known integrable discrete systems are characterized by their 'multidimensional consistency'. This means that such a system may be imposed in a consistent way in a multidimensional space. This property seems to incorporate automatically two integrability aspects of this kind of systems, in the following sense: multidimensional consistency allows one to derive algorithmically a Bäcklund transformation as well as a Lax pair for the difference equations under consideration [5–7].

Recently, Adler, Bobenko and Suris (ABS) classified the scalar lattice equations which are multidimensionally consistent and possess the symmetries of the square and the tetrahedron property, as well [8]. Subsequently, they classified the lattice equations having the consistency property in a more general framework [9].

The equations covered by the ABS classification [8] have already attracted the interest of many investigators and several results pertaining to them have already been published, including exact solutions [10, 11], Bäcklund transformations [12], symmetries [13–16] and conservation laws [17].

In this paper, we focus on the symmetry properties of the ABS equations and show how a particular class of reductions provide a natural interplay between them and certain non-autonomous systems of partial differential equations. The means to explore this link is provided by the pair of extended three-point generalized symmetries admitted by the equations of the ABS class [13].

More specifically, we study the *continuously invariant solutions* of the systems under consideration. We use the term 'continuously invariant solutions' for the solutions that remain invariant under the action of both of the extended three-point generalized symmetries admitted by the corresponding equation. We show that these solutions are determined by a system of differential–difference equations, which involves six values of the unknown function, u. The elimination of three of these values leads to an equivalent system of partial differential equations, $\Sigma[u]$, which involves the remaining values of the dependent variable.

Among the other advantages offered by the general framework of the continuously invariant solutions developed in this paper is the fact that it allows us to derive easily some of the integrability properties of $\Sigma[u]$. In particular, it enables us to construct an auto-Bäcklund transformation for this system as well as a Lax pair.

The implementation of this general framework to the equations *H1* and $Q1_{\delta=0}$ of the ABS family leads to explicit solutions, constructed using symmetry reductions of the corresponding $\Sigma[u]$ systems. These solutions are determined by quadratures from the continuous Painlevé V and VI equations, but may also be regarded as resulting from reductions which lead to discrete Painlevé equations [13, 18]. In this fashion, a new connection between discrete and continuous versions of the Painlevé equations is revealed.

Another important aspect of system $\Sigma[u]$ is that it leads to what has been termed as *generating partial differential equations*. The archetypical example of such equations is the *regular partial differential equation* (RPDE), introduced by Nijhoff, Hone and Joshi in [19]. These authors showed that the RPDE, which encodes the entire hierarchy of the Korteweg - de Vries (KdV) equation, is related to equation *H1* of the ABS family. In the present paper the above result is rederived, but by a completely different method, which also allows its immediate generalization. Specifically, we show that not only *H1*, but also *H2*, *H3* and *Q1* are related to the RPDE, and establish this relation in a systematic fashion, using the properties of the corresponding $\Sigma[u]$.

The present paper is organized as follows. In section 2, we first introduce the notation used in the sections that follow. Then, we present the main characteristics of a wider class of lattice equations containing all the members of the ABS family, along with an auto-Bäcklund transformation, $\mathbb{B}_d$, for each member of the latter.

Section 3 deals with the solutions of the equations of the ABS family which remain invariant under the action of the two extended three-point generalized symmetries admitted by the above equations. These solutions are determined by a system of differential–difference equations which we prove to be equivalent to the integrable system $\Sigma[u]$. In the same section, we prove that the class of continuously invariant solutions is closed under the Bäcklund transformation $\mathbb{B}_d$. Exploiting this result, we derive two items revealing the integrability of system $\Sigma[u]$ itself, namely an auto-Bäcklund transformation and a Lax pair.

Sections 4 and 5 contain the application of the general results of section 3 to the ABS equations *H1* and $Q1_{\delta=0}$. Specifically, we construct symmetry reductions of the corresponding $\Sigma[u]$ systems, in terms of which explicit solutions of the above equations are determined.

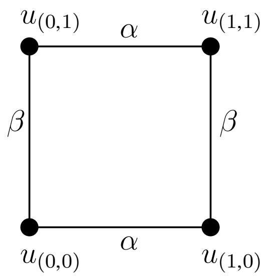

Figure 1. An elementary quadrilateral.

Section 6 deals with generating partial differential equations and the detailed analysis of system $\Sigma[u]$ corresponding to equations *H1*–*H3* and *Q1* is presented. In particular, we show that systems $\Sigma[u]$ for *H1*, *H2* and *Q1* are related, through a contact transformation, to the RPDE. Also, we derive the connection of $\Sigma[u]$ for *H3* to the RPDE. Finally, we relate our results to those of [19], where the connection of *H1*, $H3_{\delta=0}$ and $Q1_{\delta=0}$ to the RPDE was presented from a different point of view.

The concluding section contains an overall evaluation of the presented results and various perspectives.

2. Notation and the Adler–Bobenko–Suris equations

We first introduce the notation that will be used in what follows. In addition, we present those properties of the ABS equations that will be used in the following sections.

A partial difference equation is a functional relation among the values of a function $u : \mathbb{Z} \times \mathbb{Z} \to \mathbb{C}$ at various points of the lattice, which may also involve the independent variables n, m and the lattice spacings α, β (see figure 1) i.e. a relation of the form

$$\mathscr{E}(n, m, u_{(0,0)}, u_{(1,0)}, u_{(0,1)}, \ldots; \alpha, \beta) = 0. \tag{1}$$

In this relation, $u_{(i,j)}$ denotes the value of the function u at the lattice point $(n + i, m + j)$, e.g.

$$u_{(0,0)} = u(n, m), \qquad u_{(1,0)} = u(n + 1, m), \qquad u_{(0,1)} = u(n, m + 1),$$

and this is the notation that will be used for the values of the function u from now on.

The analysis of such equations is facilitated by the introduction of two translation operators acting on functions on $\mathbb{Z}^2$, defined by

$$\left(\mathscr{S}_n^{(k)} u\right)_{(0,0)} = u_{(k,0)}, \qquad \left(\mathscr{S}_m^{(k)} u\right)_{(0,0)} = u_{(0,k)}, \qquad \text{where} \quad k \in \mathbb{Z},$$

respectively.

The equations of the ABS family belong to a wider class which contains all the equations of the form

$$Q(u_{(0,0)}, u_{(1,0)}, u_{(0,1)}, u_{(1,1)}; \alpha, \beta) = 0, \tag{2}$$

where the function Q satisfies the following requirements.

(1) It does not depend explicitly on the discrete variables n, m.
(2) It is affine linear and depends explicitly on the four values of the unknown function u, i.e.

$$\partial_{u_{(i,j)}} Q(u_{(0,0)}, u_{(1,0)}, u_{(0,1)}, u_{(1,1)}; \alpha, \beta) \neq 0$$

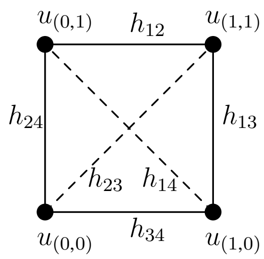

Figure 2. The elementary quadrilateral and the polynomials.

and

$$\partial^2_{u_{(i,j)}} Q(u_{(0,0)}, u_{(1,0)}, u_{(0,1)}, u_{(1,1)}; \alpha, \beta) = 0,$$

where $i, j = 0, 1$.

(3) It possesses the symmetries of the square (D_4-symmetry):

$$\begin{aligned} Q(u_{(0,0)}, u_{(1,0)}, u_{(0,1)}, u_{(1,1)}; \alpha, \beta) &= \epsilon Q(u_{(0,0)}, u_{(0,1)}, u_{(1,0)}, u_{(1,1)}; \beta, \alpha) \\ &= \sigma Q(u_{(1,0)}, u_{(0,0)}, u_{(1,1)}, u_{(0,1)}; \alpha, \beta), \end{aligned}$$

where $\epsilon = \pm 1$ and $\sigma = \pm 1$.

The affine linearity of Q implies that one can define six different polynomials in terms of the function Q [8, 9, 13], four of them assigned to the edges and the rest to the diagonals of the elementary quadrilateral where the equation is defined (see figure 2).

A polynomial h_{ij} assigned to an edge or a diagonal depends on the values of u assigned to the end points of the corresponding edge or diagonal, respectively, as illustrated in figure 2, and is defined by

$$h_{ij} = h_{ji} := Q Q_{,ij} - Q_{,i} Q_{,j}, \qquad i \neq j, \quad i, j = 1, \ldots, 4,$$

where $Q_{,i}$ denotes the derivative of Q with respect to its ith argument and $Q_{,ij}$ the second-order derivative Q with respect to its ith and jth arguments. The polynomials h_{ij} are quadratic in each one of their arguments. Moreover, the relations

$$h_{12}h_{34} = h_{13}h_{24} = h_{14}h_{23} \tag{3}$$

hold in view of the condition $Q = 0$.

On the other hand, the symmetries of the square imply the following.

(1) The polynomials on the edges have to be of the form

$$h_{ij} = \begin{cases} h(x, y; \alpha, \beta), & |i - j| = 1 \\ h(x, y; \beta, \alpha), & |i - j| = 2, \end{cases} \qquad i \neq j, \quad \{i, j\} \neq \{2, 3\}, \tag{4}$$

where h is quadratic and symmetric in its first two arguments.

(2) The two diagonal polynomials are identical and

$$h_{14} = h_{23} = G(x, y; \alpha, \beta), \tag{5}$$

where G is quadratic, symmetric in its first two arguments and symmetric in the parameters.

2.1. The Adler–Bobenko–Suris equations

In order to make our presentation self-contained, we first list all the members of the ABS classification and the notation that we will use in the following sections:

$$H1 \quad (u_{(0,0)} - u_{(1,1)})(u_{(1,0)} - u_{(0,1)}) - \alpha + \beta = 0 \tag{6}$$

$$H2 \quad (u_{(0,0)} - u_{(1,1)})(u_{(1,0)} - u_{(0,1)}) + (\beta - \alpha)(u_{(0,0)} + u_{(1,0)} + u_{(0,1)} + u_{(1,1)}) \\ - \alpha^2 + \beta^2 = 0 \tag{7}$$

$$H3 \quad \alpha(u_{(0,0)}u_{(1,0)} + u_{(0,1)}u_{(1,1)}) - \beta(u_{(0,0)}u_{(0,1)} + u_{(1,0)}u_{(1,1)}) + \delta(\alpha^2 - \beta^2) = 0 \tag{8}$$

$$Q1 \quad \alpha(u_{(0,0)} - u_{(0,1)})(u_{(1,0)} - u_{(1,1)}) - \beta(u_{(0,0)} - u_{(1,0)})(u_{(0,1)} - u_{(1,1)}) \\ + \delta^2\alpha\beta(\alpha - \beta) = 0 \tag{9}$$

$$Q2 \quad \alpha(u_{(0,0)} - u_{(0,1)})(u_{(1,0)} - u_{(1,1)}) - \beta(u_{(0,0)} - u_{(1,0)})(u_{(0,1)} - u_{(1,1)}) + \alpha\beta(\alpha - \beta) \\ \times (u_{(0,0)} + u_{(1,0)} + u_{(0,1)} + u_{(1,1)}) - \alpha\beta(\alpha - \beta)(\alpha^2 - \alpha\beta + \beta^2) = 0 \tag{10}$$

$$Q3 \quad (\beta^2 - \alpha^2)(u_{(0,0)}u_{(1,1)} + u_{(1,0)}u_{(0,1)}) + \beta(\alpha^2 - 1)(u_{(0,0)}u_{(1,0)} + u_{(0,1)}u_{(1,1)}) \\ - \alpha(\beta^2 - 1)(u_{(0,0)}u_{(0,1)} + u_{(1,0)}u_{(1,1)}) \\ - \frac{\delta^2(\alpha^2 - \beta^2)(\alpha^2 - 1)(\beta^2 - 1)}{4\alpha\beta} = 0 \tag{11}$$

$$Q4 \quad a_0 u_{(0,0)}u_{(1,0)}u_{(0,1)}u_{(1,1)} + a_1(u_{(0,0)}u_{(1,0)}u_{(0,1)} + u_{(1,0)}u_{(0,1)}u_{(1,1)} + u_{(0,1)}u_{(1,1)}u_{(0,0)} \\ + u_{(1,1)}u_{(0,0)}u_{(1,0)}) + \alpha_2(u_{(0,0)}u_{(1,1)} + u_{(1,0)}u_{(0,1)}) \\ + \bar{a}_2(u_{(0,0)}u_{(1,0)} + u_{(0,1)}u_{(1,1)}) + \tilde{a}_2(u_{(0,0)}u_{(0,1)} + u_{(1,0)}u_{(1,1)}) \\ + a_3(u_{(0,0)} + u_{(1,0)} + u_{(0,1)} + u_{(1,1)}) + a_4 = 0. \tag{12}$$

The a_i's appearing in the last equation are determined by the relations

$$a_0 = a + b, \qquad a_1 = -a\beta - b\alpha, \qquad a_2 = a\beta^2 + b\alpha^2,$$

$$\bar{a}_2 = \frac{ab(a+b)}{2(\alpha - \beta)} + a\beta^2 - \left(2\alpha^2 - \frac{g_2}{4}\right)b, \qquad \tilde{a}_2 = \frac{ab(a+b)}{2(\beta - \alpha)} + b\alpha^2 - \left(2\beta^2 - \frac{g_2}{4}\right)a,$$

$$a_3 = \frac{g_3}{2}a_0 - \frac{g_2}{4}a_1, \qquad a_4 = \frac{g_2^2}{16}a_0 - g_3 a_1,$$

where

$$a^2 = p(\alpha), \qquad b^2 = p(\beta), \qquad p(x) = 4x^3 - g_2 x - g_3.$$

The main characteristic of all of the above equations is their integrability, which is understood as they being multidimensionally consistent. From this property, it follows that [8]

(i) The polynomial related to the edges, h, can be written as

$$h(x, y; \alpha, \beta) = k(\alpha, \beta) f(x, y, \alpha),$$

where the function $k(\alpha, \beta)$ is antisymmetric, i.e.

$$k(\beta, \alpha) = -k(\alpha, \beta).$$

(ii) The discriminant

$$d := f_{,y}^2 - 2 f f_{,yy}$$

is independent of the parameters α, β.

(iii) The functions f, G and k determining the polynomials h_{ij} can be specified explicitly and, for convenience, are given in appendix A.

To the above properties of the ABS equations, one can add the following two [7], which will also be used in the symmetry analysis to be presented in the following sections.

(iv) They define their own auto-Bäcklund transformation. The latter is specified by the following relations:

$$\mathbb{B}_d(u, \tilde{u}, \lambda) := \begin{cases} Q(u_{(0,0)}, u_{(1,0)}, \tilde{u}_{(0,0)}, \tilde{u}_{(1,0)}; \alpha, \lambda) = 0 \\ Q(u_{(0,0)}, u_{(0,1)}, \tilde{u}_{(0,0)}, \tilde{u}_{(0,1)}; \beta, \lambda) = 0. \end{cases} \quad (13)$$

(v) If $\{u^0, u^1, u^2, u^{12}\}$ is a quartet of solutions related by the Bäcklund transformation $\mathbb{B}_d$, then their superposition (Bianchi diagram) is expressed by the condition

$$Q(u^0, u^1, u^2, u^{12}; \lambda_1, \lambda_2) = 0.$$

3. Symmetry reductions

In this section, we present the general framework of particular symmetry reductions of the ABS equations. More specifically, we study solutions of these equations which remain invariant under the action of both of the extended three-point generalized symmetry generators, under the assumption that the unknown function depends continuously on the lattice parameters α, β.

We first show that invariant solutions of the above kind are determined by a system of differential–difference equations. The latter turns out to be equivalent to an integrable system of partial differential equations, $\Sigma[u]$. The integrability of $\Sigma[u]$ is established by the construction of its auto-Bäcklund transformation, $\mathbb{B}_c$. This transformation provides the means for deriving a Lax pair for system $\Sigma[u]$, as well. These integrability aspects are the subject of the second part of this section.

3.1. Continuous symmetry reductions and system $\Sigma[u]$: general considerations

Let us recall that [13] equation of the ABS family

$$Q(u_{(0,0)}, u_{(1,0)}, u_{(0,1)}, u_{(1,1)}; \alpha, \beta) = 0 \quad (14)$$

admits a pair of three-point generalized symmetries generated, respectively, by the vector fields

$$\mathbf{v}_n = R(u_{(0,0)}, u_{(1,0)}, u_{(-1,0)}, \alpha)\partial_{u_{(0,0)}}, \quad (15i)$$

$$\mathbf{v}_m = R(u_{(0,0)}, u_{(0,1)}, u_{(0,-1)}, \beta)\partial_{u_{(0,0)}}. \quad (15ii)$$

It also admits a pair of extended generalized symmetries with respective generators of the vector fields:

$$\mathbf{v}_1 = A(n)R(u_{(0,0)}, u_{(1,0)}, u_{(-1,0)}, \alpha)\partial_{u_{(0,0)}} + (A(n) - A(n+1))r(\alpha)\partial_\alpha, \quad (16i)$$

$$\mathbf{v}_2 = B(m)R(u_{(0,0)}, u_{(0,1)}, u_{(0,-1)}, \beta)\partial_{u_{(0,0)}} + (B(m) - B(m+1))r(\beta)\partial_\beta, \quad (16ii)$$

where

$$R(u, x, y, \kappa) := \frac{f(u, x, \kappa)}{x - y} - \frac{1}{2} f_{,x}(u, x, \kappa) = \frac{f(u, y, \kappa)}{x - y} + \frac{1}{2} f_{,y}(u, y, \kappa), \quad (17)$$

$A(n)$, $B(m)$ are arbitrary non-constant functions of their arguments, and $r(x)$ depends on the particular equation under consideration, as specified in the following table:

Equation	*H1*	*H2*	*H3*	*Q1*	*Q2*	*Q3*	*Q4*
$r(x)$	1	1	$-\frac{x}{2}$	1	1	$-\frac{x}{2}$	$-\frac{1}{2}(4x^3 - g_2 x - g_3)^{1/2}$

The solutions of (14) that remain invariant under the action of both of the symmetry generators $\mathbf{v}_1$ and $\mathbf{v}_2$ must satisfy the invariant surface conditions

$$\frac{\partial u_{(0,0)}}{\partial \alpha} = K(n,\alpha) R(u_{(0,0)}, u_{(1,0)}, u_{(-1,0)}, \alpha), \tag{18i}$$

$$\frac{\partial u_{(0,0)}}{\partial \beta} = L(m,\beta) R(u_{(0,0)}, u_{(0,1)}, u_{(0,-1)}, \beta), \tag{18ii}$$

where

$$K(n,\alpha) := \frac{A(n)}{A(n) - A(n+1)} \frac{1}{r(\alpha)}, \qquad L(m,\beta) := \frac{B(m)}{B(m) - B(m+1)} \frac{1}{r(\beta)}. \tag{19}$$

The compatibility of equations (14) and (18) is expressed by the conditions

$$\mathrm{D}_\alpha(Q(u_{(0,0)}, u_{(1,0)}, u_{(0,1)}, u_{(1,1)}; \alpha, \beta)) = 0, \tag{20i}$$

$$\mathrm{D}_\beta(Q(u_{(0,0)}, u_{(1,0)}, u_{(0,1)}, u_{(1,1)}; \alpha, \beta)) = 0, \tag{20ii}$$

$$\partial_\beta(\partial_\alpha u_{(0,0)}) = \partial_\alpha(\partial_\beta u_{(0,0)}), \tag{20iii}$$

where D_α and D_β denote the total derivative operators with respect to α and β, respectively, i.e.

$$\mathrm{D}_\alpha := \partial_\alpha + \sum_{i,j=0}^{1} \frac{\partial u_{(i,j)}}{\partial \alpha} \partial_{u_{(i,j)}}, \qquad \mathrm{D}_\beta := \partial_\beta + \sum_{i,j=0}^{1} \frac{\partial u_{(i,j)}}{\partial \beta} \partial_{u_{(i,j)}}.$$

Written out explicitly, by using the expressions for $Q_{,\alpha}$ and $Q_{,\beta}$ following from the determining equations for the symmetry generators $\mathbf{v}_1$ and $\mathbf{v}_2$, respectively, conditions (20i) and (20ii) imply that $A(n)$ and $B(m)$ must be affine linear. Without loss of generality, we choose them to read as follows:

$$A(n) = n, \qquad B(m) = m.$$

Condition (20iii), on the other hand, imposes no further restrictions, because it holds identically. This follows from the fact that the commutator of the two symmetry generators $\mathbf{v}_n$, $\mathbf{v}_m$ produces a trivial generalized symmetry [7].

Thus, the solutions of the ABS equations which are invariant under the action of both $\mathbf{v}_1$ and $\mathbf{v}_2$ are determined by the differential–difference system

$$Q(u_{(0,0)}, u_{(1,0)}, u_{(0,1)}, u_{(1,1)}; \alpha, \beta) = 0, \tag{21i}$$

$$r(\alpha) \frac{\partial u_{(0,0)}}{\partial \alpha} + n R(u_{(0,0)}, u_{(1,0)}, u_{(-1,0)}, \alpha) = 0, \tag{21ii}$$

$$r(\beta) \frac{\partial u_{(0,0)}}{\partial \beta} + m R(u_{(0,0)}, u_{(0,1)}, u_{(0,-1)}, \beta) = 0. \tag{21iii}$$

These solutions will be referred to as *continuously invariant solutions*.

System (21) involves the values of the unknown function u at six different points of the lattice. One could eliminate any three of these values and get an equivalent system of *partial differential equations* involving the remaining ones. We choose to eliminate the values $u_{(-1,0)}$, $u_{(0,-1)}$ and $u_{(1,1)}$, and this leads to the following result.

Proposition 3.1. *Every continuous invariant solution is determined by the system of partial differential equations*

$$\frac{\partial u_{(1,0)}}{\partial \beta} = \frac{G(u_{(1,0)}, u_{(0,1)})}{k(\alpha,\beta) f(u_{(0,0)}, u_{(0,1)}, \beta)} \frac{\partial u_{(0,0)}}{\partial \beta} + \frac{m f(u_{(0,0)}, u_{(0,1)}, \beta)}{2r(\beta)k(\alpha,\beta)} \partial_{u_{(0,1)}} \left(\frac{G(u_{(1,0)}, u_{(0,1)})}{f(u_{(0,0)}, u_{(0,1)}, \beta)} \right),$$

$$\frac{\partial u_{(0,1)}}{\partial \alpha} = \frac{G(u_{(1,0)}, u_{(0,1)})}{k(\beta,\alpha) f(u_{(0,0)}, u_{(1,0)}, \alpha)} \frac{\partial u_{(0,0)}}{\partial \alpha} + \frac{n f(u_{(0,0)}, u_{(1,0)}, \alpha)}{2r(\alpha)k(\beta,\alpha)} \partial_{u_{(1,0)}} \left(\frac{G(u_{(1,0)}, u_{(0,1)})}{f(u_{(0,0)}, u_{(1,0)}, \alpha)} \right),$$

$$\frac{\partial^2 u_{(0,0)}}{\partial \alpha \partial \beta} = A_1 \frac{\partial u_{(0,0)}}{\partial \alpha} \frac{\partial u_{(0,0)}}{\partial \beta} + \frac{f(u_{(0,0)}, u_{(1,0)}, \alpha)}{2k(\alpha,\beta)} \left(\frac{m}{r(\beta)} A_2 \frac{\partial u_{(0,0)}}{\partial \alpha} + \frac{n}{r(\alpha)} A_3 \frac{\partial u_{(0,0)}}{\partial \beta} \right) + \frac{nm f(u_{(0,0)}, u_{(1,0)}, \alpha)}{4r(\alpha)r(\beta)k(\alpha,\beta)} A_4,$$

where

$$A_1 = \left(\frac{f_{,u_{(0,0)}}(u_{(0,0)}, u_{(1,0)}, \alpha)}{f(u_{(0,0)}, u_{(1,0)}, \alpha)} - \frac{f(u_{(0,0)}, u_{(1,0)}, \alpha)}{k(\alpha,\beta) f(u_{(0,0)}, u_{(0,1)}, \beta)} \partial_{u_{(1,0)}} \left(\frac{G(u_{(1,0)}, u_{(0,1)})}{f(u_{(0,0)}, u_{(1,0)}, \alpha)} \right) \right),$$

$$A_2 = \partial_{u_{(1,0)}} \left(\frac{f_{,u_{(0,1)}}(u_{(0,0)}, u_{(0,1)}, \beta)}{f(u_{(0,0)}, u_{(0,1)}, \beta)} \frac{G(u_{(1,0)}, u_{(0,1)})}{f(u_{(0,0)}, u_{(1,0)}, \alpha)} - \frac{G_{,u_{(0,1)}}(u_{(1,0)}, u_{(0,1)})}{f(u_{(0,0)}, u_{(1,0)}, \alpha)} \right),$$

$$A_3 = k(\alpha,\beta) \partial_{u_{(1,0)}} (\ln f(u_{(0,0)}, u_{(1,0)}, \alpha)) - \frac{f(u_{(0,0)}, u_{(1,0)}, \alpha)}{f(u_{(0,0)}, u_{(0,1)}, \beta)} \partial_{u_{(1,0)}} \left(\frac{G_{,u_{(1,0)}}(u_{(1,0)}, u_{(0,1)})}{f(u_{(0,0)}, u_{(1,0)}, \alpha)} \right) + \frac{G(u_{(1,0)}, u_{(0,1)})}{f(u_{(0,0)}, u_{(0,1)}, \beta)} \partial^2_{u_{(1,0)}} (\ln f(u_{(0,0)}, u_{(1,0)}, \alpha))$$

and

$$A_4 = \partial_{u_{(1,0)}} \partial_{u_{(0,1)}} \left(\frac{f_{,u_{(1,0)}}(u_{(0,0)}, u_{(1,0)}, \alpha) G(u_{(1,0)}, u_{(0,1)})}{f(u_{(0,0)}, u_{(1,0)}, \alpha)} - G_{,u_{(1,0)}}(u_{(1,0)}, u_{(0,1)}) \right) + \partial_{u_{(0,1)}} (\ln f(u_{(0,0)}, u_{(0,1)}, \beta)) f(u_{(0,0)}, u_{(1,0)}, \alpha) \partial_{u_{(1,0)}} \left(\frac{G_{,u_{(1,0)}}(u_{(1,0)}, u_{(0,1)})}{f(u_{(0,0)}, u_{(1,0)}, \alpha)} \right) - \partial_{u_{(0,1)}} (\ln f(u_{(0,0)}, u_{(0,1)}, \beta)) \partial^2_{u_{(1,0)}} (\ln f(u_{(0,0)}, u_{(1,0)}, \alpha)) G(u_{(1,0)}, u_{(0,1)}).$$

The above system, which will be denoted by $\Sigma(u_{(0,0)}, u_{(1,0)}, u_{(0,1)}; \alpha, \beta; n, m)$*, or, simply* $\Sigma[u]$*, is symmetric:*

$$\Sigma(u_{(0,0)}, u_{(0,1)}, u_{(1,0)}; \beta, \alpha; m, n) = \Sigma(u_{(0,0)}, u_{(1,0)}, u_{(0,1)}; \alpha, \beta; n, m).$$

Proof. The first equation of system $\Sigma[u]$ results by eliminating the values $u_{(0,-1)}$ and $u_{(1,-1)}$ from equation (21*iii*) and

$$Q(u_{(0,-1)}, u_{(1,-1)}, u_{(0,0)}, u_{(1,0)}; \alpha, \beta) = 0.$$

Using the affine linearity and the symmetries of Q, the last equation can be written as

$$u_{(1,-1)} = -\frac{Q_{,u_{(0,1)}} u_{(0,-1)} + Q}{Q_{,u_{(0,1)}u_{(1,1)}} u_{(0,-1)} + Q_{,u_{(1,1)}}}, \tag{22}$$

where the arguments of $Q(u_{(0,0)}, u_{(1,0)}, u_{(0,1)}, u_{(1,1)}; \alpha, \beta)$ have been omitted and Q and its derivatives are understood to be evaluated at $u_{(0,1)} = u_{(1,1)} = 0$.

We now solve (21iii) and its shift in the n direction with respect to $u_{(0,-1)}$ and $u_{(1,-1)}$, respectively, and substitute the results into equation (22). The resulting equation, combined with the relations

$$Q^2_{,u_{(1,1)}} = \frac{f(u_{(0,0)}, u_{(1,0)}, \alpha)G(u_{(1,0)}, u_{(0,1)})}{f(u_{(1,0)}, u_{(1,1)}, \beta)},$$

$$\partial_{u_{(0,1)}} Q^2_{,u_{(1,1)}} = \frac{k(\alpha,\beta) f(u_{(0,0)}, u_{(1,0)}, \alpha)\big(G_{,u_{(0,1)}}(u_{(1,0)}, u_{(0,1)}) - f_{,u_{(1,1)}}(u_{(1,0)}, u_{(1,1)}, \beta)\big)}{f(u_{(1,0)}, u_{(1,1)}, \beta)},$$

which hold in view of $Q = 0$, yields the first member of $\Sigma[u]$.

The second equation of $\Sigma[u]$ results in a similar manner. It is also easily verified that the first two equations of $\Sigma[u]$ are symmetric, i.e. the one is mapped to the other under interchanges

$$u_{(1,0)} \longleftrightarrow u_{(0,1)}, \qquad \alpha \longleftrightarrow \beta, \qquad n \longleftrightarrow m. \tag{23}$$

In order to obtain the third member of $\Sigma[u]$, one first solves the second equation of $\Sigma[u]$ and its shift in the n direction for $\partial_\beta u_{(1,0)}$ and $\partial_\beta u_{(-1,0)}$. One then substitutes the result into the derivative of equation (21ii) with respect to β. From the resulting equation, one arrives at the third member of $\Sigma[u]$ by using the expressions for $u_{(1,0)} - u_{(-1,0)}$, $G(u_{(-1,0)}, u_{(0,1)})$ and its derivatives provided by equation (21ii) and the relation[1]

$$G_{,u_{(1,0)}}(u_{(1,0)}, u_{(0,1)}) + G_{,u_{(-1,0)}}(u_{(-1,0)}, u_{(0,1)}) = 2\frac{G(u_{(1,0)}, u_{(0,1)}) - G(u_{(-1,0)}, u_{(0,1)})}{u_{(1,0)} - u_{(-1,0)}},$$

and its differential consequences, respectively.

Finally, differentiating equation (21iii) with respect to α and following an analogous procedure, one arrives at an expression which is identical to the third member of $\Sigma[u]$ under the mapping (23). □

As already noted, one may choose to eliminate any other triad of the values of u involved in equations (21). In this fashion, compatible systems of partial differential equations can be constructed involving the triplets $(u_{(0,0)}, u_{(-1,0)}, u_{(0,1)})$, $(u_{(0,0)}, u_{(1,0)}, u_{(0,-1)})$ and $(u_{(0,0)}, u_{(-1,0)}, u_{(0,-1)})$, respectively. It turns out that the corresponding systems are given by

$$\Sigma(u_{(0,0)}, u_{(-1,0)}, u_{(0,1)}; \alpha, \beta; -n, m), \qquad \Sigma(u_{(0,0)}, u_{(1,0)}, u_{(0,-1)}; \alpha, \beta; n, -m)$$

and

$$\Sigma(u_{(0,0)}, u_{(-1,0)}, u_{(0,-1)}; \alpha, \beta; -n, -m),$$

respectively.

System $\Sigma(u_{(0,0)}, u_{(1,0)}, u_{(0,1)}; \alpha, \beta; n, m)$ and the last three are compatible, in the following sense (cf Figure 3). If we eliminate $\partial_\beta u_{(1,0)}$ (respectively $\partial_\beta u_{(-1,0)}$) from systems $\Sigma(n, m)$ and $\Sigma(n, -m)$ (respectively $\Sigma(-n, m)$ and $\Sigma(-n, -m)$), then we will end up with (21iii). On the other hand, the elimination of $\partial_\alpha u_{(0,1)}$ and $\partial_\alpha u_{(0,-1)}$ from systems $\Sigma(n, m)$, $\Sigma(-n, m)$ and $\Sigma(n, -m)$, $\Sigma(-n, -m)$, respectively, leads to (21ii). Finally, the elimination of $\partial_\alpha \partial_\beta u_{(0,0)}$ from any two of the four $\Sigma[u]$'s mentioned above results in (21i).

[1] This relation holds identically, i.e. without taking into account the equation $Q = 0$; thus, we can differentiate it assuming that the corresponding values of u are independent.

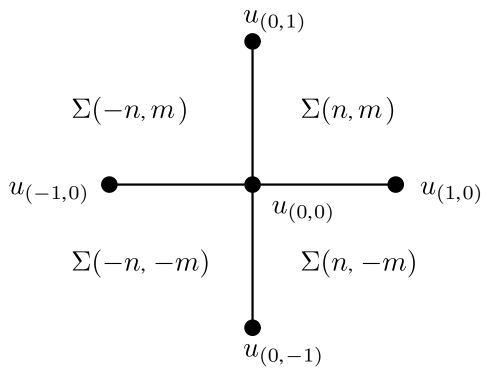

Figure 3. The values of u and the compatible systems Σ.

3.2. Integrability of system $\Sigma[u]$

We have already characterized system $\Sigma[u]$ as integrable. To support this characterization, in the present subsection, we construct an auto-Bäcklund transformation and a Lax pair for the above system.

To this end, let it first be noted that the fact that every equation of the ABS class

$$Q(u_{(0,0)}, u_{(1,0)}, u_{(0,1)}, u_{(1,1)}; \alpha, \beta) = 0 \tag{24}$$

admits generalized symmetries and extended generalized symmetries with generators of the vector fields given in (15) and (16), respectively, has the following consequences.

(1) The vector fields

$$\tilde{\mathbf{v}}_n = R(u_{(0,0)}, u_{(1,0)}, u_{(-1,0)}, \alpha)\partial_{u_{(0,0)}} + R(\tilde{u}_{(0,0)}, \tilde{u}_{(1,0)}, \tilde{u}_{(-1,0)}, \alpha)\partial_{\tilde{u}_{(0,0)}}$$

and

$$\tilde{\mathbf{v}}_1 = A(n)\tilde{\mathbf{v}}_n + (A(n) - A(n+1))r(\alpha)\partial_\alpha$$

are symmetry generators of the first of the equations making up the auto-Bäcklund transformation $\mathbb{B}_d$, i.e. of $Q(u_{(0,0)}, u_{(1,0)}, \tilde{u}_{(0,0)}, \tilde{u}_{(1,0)}; \alpha, \lambda) = 0$. Therefore, relations

$$\tilde{\mathbf{v}}_n^{(1)}(Q(u_{(0,0)}, u_{(1,0)}, \tilde{u}_{(0,0)}, \tilde{u}_{(1,0)}; \alpha, \lambda)) = 0,$$
$$\tilde{\mathbf{v}}_1^{(1)}(Q(u_{(0,0)}, u_{(1,0)}, \tilde{u}_{(0,0)}, \tilde{u}_{(1,0)}; \alpha, \lambda)) = 0$$

hold in view of $Q(u_{(0,0)}, u_{(1,0)}, \tilde{u}_{(0,0)}, \tilde{u}_{(1,0)}; \alpha, \lambda) = 0$.

(2) The vector fields

$$\tilde{\mathbf{v}}_m = R(u_{(0,0)}, u_{(0,1)}, u_{(0,-1)}, \beta)\partial_{u_{(0,0)}} + R(\tilde{u}_{(0,0)}, \tilde{u}_{(0,1)}, \tilde{u}_{(0,-1)}, \beta)\partial_{\tilde{u}_{(0,0)}},$$
$$\tilde{\mathbf{v}}_2 = B(m)\mathbf{v}_m + (B(m) - B(m+1))r(\beta)\partial_\beta,$$

are symmetry generators of the second of the equations of the auto-Bäcklund transformation. As a result, the pair of relations

$$\tilde{\mathbf{v}}_m^{(1)}(Q(u_{(0,0)}, u_{(0,1)}, \tilde{u}_{(0,0)}, \tilde{u}_{(0,1)}; \beta, \lambda)) = 0,$$
$$\tilde{\mathbf{v}}_2^{(1)}(Q(u_{(0,0)}, u_{(0,1)}, \tilde{u}_{(0,0)}, \tilde{u}_{(0,1)}; \beta, \lambda)) = 0$$

hold in view of $Q(u_{(0,0)}, u_{(0,1)}, \tilde{u}_{(0,0)}, \tilde{u}_{(0,1)}; \beta, \lambda) = 0$.

Using the above observations, one may prove the following proposition.

Proposition 3.2. *The auto-Bäcklund transformation* $\mathbb{B}_d(u, \tilde{u}, \lambda)$ *maps a continuously invariant solution u to another solution* $\tilde{u}$ *of the same kind.*

Proof. It is given in the appendix B. □

An immediate consequence of the this result is described in the following proposition.

Proposition 3.3. *If u is a continuously invariant solution, then the system*

$$Q(u_{(0,0)}, u_{(1,0)}, \tilde{u}_{(0,0)}, \tilde{u}_{(1,0)}; \alpha, \lambda) = 0, \tag{25i}$$

$$Q(u_{(0,0)}, u_{(0,1)}, \tilde{u}_{(0,0)}, \tilde{u}_{(0,1)}; \beta, \lambda) = 0, \tag{25ii}$$

$$\frac{\partial \tilde{u}_{(0,0)}}{\partial \alpha} = \frac{1}{k(\alpha,\lambda)}\left(-\frac{\partial u_{(0,0)}}{\partial \alpha} + \frac{n}{2r(\alpha)} f_{,u_{(1,0)}}(u_{(0,0)}, u_{(1,0)}, \alpha)\right)\frac{G(u_{(1,0)}, \tilde{u}_{(0,0)}, \alpha, \lambda)}{f(u_{(0,0)}, u_{(1,0)}, \alpha)} - \frac{n}{2k(\alpha,\lambda)r(\alpha)} G_{,u_{(1,0)}}(u_{(1,0)}, \tilde{u}_{(0,0)}, \alpha, \lambda), \tag{25iii}$$

$$\frac{\partial \tilde{u}_{(0,0)}}{\partial \beta} = \frac{1}{k(\beta,\lambda)}\left(-\frac{\partial u_{(0,0)}}{\partial \beta} + \frac{m}{2r(\beta)} f_{,u_{(0,1)}}(u_{(0,0)}, u_{(0,1)}, \beta)\right)\frac{G(u_{(0,1)}, \tilde{u}_{(0,0)}, \beta, \lambda)}{f(u_{(0,0)}, u_{(0,1)}, \beta)} - \frac{m}{2k(\beta,\lambda)r(\beta)} G_{,u_{(0,1)}}(u_{(0,1)}, \tilde{u}_{(0,0)}, \beta, \lambda) \tag{25iv}$$

defines a new solution $\tilde{u}$ *of the same kind and conversely.*

The above system, which will be denoted as $\mathbb{B}_c(u, \tilde{u}, \lambda)$*, is symmetric*

$$\mathbb{B}_c(\tilde{u}, u, \lambda) = \mathbb{B}_c(u, \tilde{u}, \lambda),$$

and defines an auto-Bäcklund transformation of system $\Sigma[u]$.

Proof. It is given in appendix C. □

Remark 3.1.

(i) The second pair of equations of system $\mathbb{B}_c(u, \tilde{u}, \lambda)$ follows from the first two equations of $\Sigma[u]$, via the substitutions

$$u_{(1,0)} \longrightarrow \tilde{u}_{(0,0)}, \qquad \alpha \longrightarrow \lambda$$

and

$$u_{(0,1)} \longrightarrow \tilde{u}_{(0,0)}, \qquad \beta \longrightarrow \lambda,$$

respectively.

(ii) The superposition principle of $\mathbb{B}_d(u, \tilde{u}, \lambda)$ implies the corresponding one for $\mathbb{B}_c(u, \tilde{u}, \lambda)$:

$$Q\left(u^0_{(0,0)}, u^1_{(0,0)}, u^2_{(0,0)}, u^{12}_{(0,0)}; \lambda_1, \lambda_2\right) = 0,$$
$$Q\left(u^0_{(1,0)}, u^1_{(1,0)}, u^2_{(1,0)}, u^{12}_{(1,0)}; \lambda_1, \lambda_2\right) = 0,$$
$$Q\left(u^0_{(0,1)}, u^1_{(0,1)}, u^2_{(0,1)}, u^{12}_{(0,1)}; \lambda_1, \lambda_2\right) = 0.$$

Finally, let us consider the following pair of equations:

$$\Phi_{,\alpha} = \frac{1}{k(\alpha,\lambda)}\begin{pmatrix} -\frac{1}{2}A_{,\tilde{u}_{(0,0)}} & -\frac{1}{2}A_{,\tilde{u}_{(0,0)}\tilde{u}_{(0,0)}} \\ A & \frac{1}{2}A_{,\tilde{u}_{(0,0)}} \end{pmatrix}\Phi, \tag{26i}$$

$$\Phi_{,\beta} = \frac{1}{k(\beta,\lambda)}\begin{pmatrix} -\frac{1}{2}B_{,\tilde{u}_{(0,0)}} & -\frac{1}{2}B_{,\tilde{u}_{(0,0)}\tilde{u}_{(0,0)}} \\ B & \frac{1}{2}B_{,\tilde{u}_{(0,0)}} \end{pmatrix}\Phi, \tag{26ii}$$

where

$$A := \left(-\frac{\partial u_{(0,0)}}{\partial \alpha} + \frac{n}{2r(\alpha)} f_{,u_{(1,0)}}(u_{(0,0)}, u_{(1,0)}, \alpha) \right) \frac{G(u_{(1,0)}, \tilde{u}_{(0,0)}, \alpha, \lambda)}{f(u_{(0,0)}, u_{(1,0)}, \alpha)} - \frac{n}{2r(\alpha)} G_{,u_{(1,0)}}(u_{(1,0)}, \tilde{u}_{(0,0)}, \alpha, \lambda), \tag{27i}$$

$$B := \left(-\frac{\partial u_{(0,0)}}{\partial \beta} + \frac{m}{2r(\beta)} f_{,u_{(0,1)}}(u_{(0,0)}, u_{(0,1)}, \beta) \right) \frac{G(u_{(0,1)}, \tilde{u}_{(0,0)}, \beta, \lambda)}{f(u_{(0,0)}, u_{(0,1)}, \beta)} - \frac{m}{2r(\beta)} G_{,u_{(0,1)}}(u_{(0,1)}, \tilde{u}_{(0,0)}, \beta, \lambda), \tag{27ii}$$

and A, B and their derivatives are evaluated at $\tilde{u}_{(0,0)} = 0$. Equations (26) constitute a Lax pair for system $\Sigma[u]$. One arrives at this result, essentially, by the inverse of the procedure presented by Crampin in [20]. In any case, it can be easily verified directly by considering each of the ABS equations separately.

4. Continuous-invariant solutions of the discrete potential KdV equation

In the last section, we established the general framework for the special reductions of the ABS equations leading to what we called continuously invariant solutions. In the present section, the above results are applied to equation *H1*. The latter will also be referred to as *discrete potential KdV*, in compliance with the terminology adopted in [21] (cf also [2, 3]).

System $\Sigma[u]$ corresponding to *H1* is made up of the equations

$$\frac{\partial u_1}{\partial \beta} = \frac{u_1 - u_2}{\alpha - \beta} \left(m - (u_1 - u_2) \frac{\partial u}{\partial \beta} \right), \tag{28i}$$

$$\frac{\partial u_2}{\partial \alpha} = \frac{u_1 - u_2}{\alpha - \beta} \left(n + (u_1 - u_2) \frac{\partial u}{\partial \alpha} \right), \tag{28ii}$$

$$\frac{\partial^2 u}{\partial \alpha \partial \beta} = \frac{1}{\alpha - \beta} \left(2(u_1 - u_2) \frac{\partial u}{\partial \alpha} \frac{\partial u}{\partial \beta} + n \frac{\partial u}{\partial \beta} - m \frac{\partial u}{\partial \alpha} \right), \tag{28iii}$$

where

$$u = u_{(0,0)}, \qquad u_1 = u_{(1,0)}, \qquad u_2 = u_{(0,1)}.$$

Obviously, the nonlinear system (28) is very hard to solve. However, whole families of solutions can be obtained, in a systematic way, via symmetry analysis. In what follows, we construct multiparameter families of solutions of the above system, using its Lie point symmetries.

System (28) admits a five-dimensional group of point symmetries generated by the vector fields [24]

$$\mathbf{w}_1 = \partial_\alpha + \partial_\beta, \qquad \mathbf{w}_2 = \alpha \partial_\alpha + \beta \partial_\beta + u \partial_u,$$
$$\mathbf{w}_3 = \partial_u, \qquad \mathbf{w}_4 = \partial_{u_1} + \partial_{u_2}, \qquad \mathbf{w}_5 = u \partial_u - u_1 \partial_{u_1} - u_2 \partial_{u_2}.$$

It will be shown that similarity solutions corresponding to the above group of symmetries are determined by solutions of the Painlevé V and VI equations [24]. For easy reference, we note that the latter equations are given by

$$G'' = \left(\frac{1}{2G} + \frac{1}{G-1} \right) G'^2 - \frac{1}{y} G' + \mathfrak{a} \frac{G(G-1)^2}{y^2} + \mathfrak{b} \frac{(G-1)^2}{y^2 G} + \mathfrak{c} \frac{G}{y} + \mathfrak{d} \frac{G(G+1)}{G-1} \tag{29}$$

and

$$G'' = \frac{1}{2}\left(\frac{1}{G} + \frac{1}{G-1} + \frac{1}{G-y}\right)G'^2 - \left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{G-y}\right)G'$$
$$+ \frac{G(G-1)(G-y)}{y^2(y-1)^2}\left(\mathfrak{a} + \mathfrak{b}\frac{y}{G^2} + \mathfrak{c}\frac{y-1}{(G-1)^2} + \mathfrak{d}\frac{y(y-1)}{(G-y)^2}\right), \qquad (30)$$

which will be denoted by $\mathscr{P}_{\mathrm{V}}(y, G(y); \mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d})$ and $\mathscr{P}_{\mathrm{VI}}(y, G(y); \mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d})$, respectively.

For the same reason, we list here the symmetry generators of the discrete potential KdV equation [13, 15], which will be used extensively in the rest of this section.

- Point symmetries

$$\mathbf{x}_1 = \partial_{u_{(0,0)}}, \qquad \mathbf{x}_2 = (-1)^{n+m}\partial_{u_{(0,0)}}, \qquad \mathbf{x}_3 = (-1)^{n+m}u_{(0,0)}\partial_{u_{(0,0)}},$$
$$\mathbf{x}_4 = u_{(0,0)}\partial_{u_{(0,0)}} + 2\alpha\partial_\alpha + 2\beta\partial_\beta, \qquad \mathbf{x}_5 = \partial_\alpha + \partial_\beta.$$

- Generalized symmetries

$$\mathbf{v}_1 = \frac{1}{u_{(1,0)} - u_{(-1,0)}}\partial_{u_{(0,0)}}, \qquad \mathbf{v}_2 = n\mathbf{v}_1 + \frac{u_{(0,0)}}{2(\alpha-\beta)}\partial_{u_{(0,0)}},$$
$$\mathbf{v}_3 = \frac{1}{u_{(0,1)} - u_{(0,-1)}}\partial_{u_{(0,0)}}, \qquad \mathbf{v}_4 = m\mathbf{v}_3 - \frac{u_{(0,0)}}{2(\alpha-\beta)}\partial_{u_{(0,0)}},$$
$$\mathbf{v}_5 = n\mathbf{v}_1 - \partial_\alpha, \qquad \mathbf{v}_6 = m\mathbf{v}_3 - \partial_\beta.$$

4.1. Solutions related to Painlevé V

We first consider solutions of system (28) that are invariant under the action of the symmetry generator

$$\mathbf{w}_1 + 2\mu\mathbf{w}_5 = \partial_\alpha + \partial_\beta + 2\mu\big(u\partial_u - u_1\partial_{u_1} - u_2\partial_{u_2}\big), \qquad \mu \in \mathbb{R} - \{0\}, \qquad (31)$$

where μ may depend on n, m. Such solutions have to satisfy the differential equations

$$u_{,\alpha} + u_{,\beta} = 2\mu u, \qquad u_{1,\alpha} + u_{1,\beta} = -2\mu u_1, \qquad u_{2,\alpha} + u_{2,\beta} = -2\mu u_2.$$

This implies that u, u_1 and u_2 are given by

$$u(\alpha,\beta) = T_{n,m}(y)\exp(\mu z),$$
$$u_1(\alpha,\beta) = T_{n+1,m}(y)\exp(-\mu z), \qquad (32)$$
$$u_2(\alpha,\beta) = T_{n,m+1}(y)\exp(-\mu z),$$

where

$$y = \alpha - \beta, \qquad z = \alpha + \beta,$$

and $T_{i,j}(y)$ are arbitrary functions.

Substitution of these forms into (28) yields the following system of ordinary differential equations:

$$y\{T'_{n+1,m}(y) + \mu T_{n+1,m}(y)\} = -(m + \mathscr{A}^-\mathscr{B})\mathscr{B},$$
$$y\{T'_{n,m+1}(y) - \mu T_{n,m+1}(y)\} = (n + \mathscr{A}^+\mathscr{B})\mathscr{B}, \qquad (33)$$
$$yT''_{n,m}(y) - (n+m)T'_{n,m}(y) = 2\mathscr{A}^+\mathscr{A}^-\mathscr{B} - (n - m - \mu y)\mu T_{n,m}(y),$$

where

$$\mathscr{A}^\pm := T'_{n,m}(y) \pm \mu T_{n,m}(y), \qquad \mathscr{B} := T_{n+1,m}(y) - T_{n,m+1}(y),$$

and the prime denotes differentiation with respect to y. The analysis of the latter system can be summarized as follows.

Starting from (33) and using differentiation and elimination, one arrives at a fourth-order ordinary differential equation for $T_{n,m}(y)$, which is omitted because of its length. The order of the latter differential equation is reduced by 1, using the quadrature

$$\frac{\mathrm{d}}{\mathrm{d}y}\ln(T_{n,m}(y)) = \mu\frac{1+G_{n,m}(y)}{1-G_{n,m}(y)}. \tag{34i}$$

Finally, the resulting third-order equation can be integrated once to yield

$$\mathscr{P}_{\mathrm{V}}\left(y, G_{n,m}(y); \frac{n^2}{2}, -\frac{m^2}{2}, \lambda, -2\mu^2\right), \tag{34ii}$$

where λ is a constant of integration and may depend on the parameters n, m.

Returning to system (33) and using relations (34), one finds the following expressions for the functions $T_{n+1,m}(y)$, $T_{n,m+1}(y)$:

$$T_{n+1,m}(y) = \frac{yG'_{n,m}(y) + nG^2_{n,m}(y) - (2n+1+\kappa-2\mu y)\, G_{n,m}(y) + n + \kappa + 1}{4\mu(1-G_{n,m}(y))T_{n,m}(y)}, \tag{35i}$$

$$T_{n,m+1}(y) = \frac{yG'_{n,m}(y) - (m+\kappa+1)G^2_{n,m}(y) + (2m+1+\kappa+2\mu y)G_{n,m}(y) - m}{4\mu(1-G_{n,m}(y))G_{n,m}(y)T_{n,m}(y)}, \tag{35ii}$$

where $\kappa = \lambda/(2\mu)$.

In order to satisfy the discrete potential KdV equation, the functions u, u_1 and u_2, given by (32), (34) and (35), respectively, must be related by shifting appropriately n and m, i.e.

$$u_1 = \mathscr{S}_n(u), \qquad u_2 = \mathscr{S}_m(u).$$

The above conditions imply that parameter μ must have the form

$$\mu = (-1)^{n+m}\tau, \qquad \tau \in \mathbb{R},$$

and functions $T_{i,j}(y)$ have to satisfy the following relations:

$$T_{n+1,m}(y) = \mathscr{S}_n(T_{n,m}(y)), \qquad T_{n,m+1}(y) = \mathscr{S}_m(T_{n,m}(y)).$$

The combination of these conditions with equations (34) and (35) leads to the following restriction on parameter λ. It must be of the form

$$\lambda = \rho - (-1)^{n+m}\tau, \qquad \rho \in \mathbb{R}.$$

Recapitulating, we can state that the discrete potential KdV equation admits continuously invariant solutions of the form

$$u_{(0,0)} = T_{n,m}(\alpha-\beta)\exp[(-1)^{n+m}\tau \times (\alpha+\beta)], \qquad \tau \in \mathbb{R}, \tag{36i}$$

where $T_{n,m}(y)$ is given by the quadrature

$$\frac{\mathrm{d}}{\mathrm{d}y}\ln(T_{n,m}(y)) = (-1)^{n+m}\tau\frac{1+G_{n,m}(y)}{1-G_{n,m}(y)}, \tag{36ii}$$

with $G_{n,m}(y)$ being a solution of the Painlevé equation

$$\mathscr{P}_{\mathrm{V}}\left(y, G_{n,m}(y); \frac{n^2}{2}, -\frac{m^2}{2}, \rho-(-1)^{n+m}\tau, -2\tau^2\right), \qquad \rho \in \mathbb{R}. \tag{36iii}$$

Remark 4.1. The solutions of the discrete potential KdV equation just constructed can also be considered as being derived from what is referred to as the *asymmetric, alternate* *discrete Painlevé II equation*. This observation results from the following considerations. By construction, any solution u of the class derived above satisfies the differential equation

$$\partial_\alpha u_{(0,0)} + \partial_\beta u_{(0,0)} = 2\tau(-1)^{n+m} u_{(0,0)} \tag{37}$$

as well as the invariant surface conditions (21*ii*) and (21*iii*), i.e.

$$\partial_\alpha u_{(0,0)} + \frac{n}{u_{(1,0)} - u_{(-1,0)}} = 0, \qquad \partial_\beta u_{(0,0)} + \frac{m}{u_{(0,1)} - u_{(0,-1)}} = 0. \tag{38}$$

Elimination of the derivatives of $u_{(0,0)}$ from (37) and (38) leads to

$$\frac{n}{u_{(1,0)} - u_{(-1,0)}} + \frac{m}{u_{(0,1)} - u_{(0,-1)}} + 2\tau(-1)^{n+m} u_{(0,0)} = 0. \tag{39}$$

This, however, is the invariant surface condition for solutions of the discrete potential KdV that remain invariant under the action of the symmetry generator $\mathbf{v}_2 + \mathbf{v}_4 + 2\tau\mathbf{x}_3$. As shown in [13], this class of group-invariant solutions is determined by solutions of the asymmetric, alternate discrete Painlevé II.

4.2. Solutions related to Painlevé VI

The solutions of system (28) which remain invariant under the action of the symmetry generator

$$\mathbf{w}_2 + (2\mu - 1)\mathbf{w}_5 = \alpha\partial_\alpha + \beta\partial_\beta + 2\mu u\partial_u + (1 - 2\mu)\left(u_1\partial_{u_1} + u_2\partial_{u_2}\right), \quad \mu \in \mathbb{R} - \{0\}, \tag{40}$$

where μ may depend on n and m, must satisfy the differential equations

$$\alpha u_{,\alpha} + \beta u_{,\beta} = 2\mu u,$$
$$\alpha u_{1,\alpha} + \beta u_{1,\beta} = (1 - 2\mu)u_1,$$
$$\alpha u_{2,\alpha} + \beta u_{2,\beta} = (1 - 2\mu)u_2.$$

Hence, these invariant solutions must have the form

$$\begin{aligned} u(\alpha, \beta) &= S_{n,m}(y) z^{\mu}, \\ u_1(\alpha, \beta) &= S_{n+1,m}(y) z^{1/2-\mu}, \\ u_2(\alpha, \beta) &= S_{n,m+1}(y) z^{1/2-\mu}, \end{aligned} \tag{41}$$

where

$$y = \frac{\alpha}{\beta}, \qquad z = \alpha\beta.$$

Substitution of the above expressions into system (28) leads to the system of ordinary differential equations:

$$(1 - y)\,\{2yS'_{n+1,m}(y) + (2\mu - 1)S_{n+1,m}(y)\} = 2\{m + \sqrt{y}\mathscr{A}^-\mathscr{B}\}\mathscr{B},$$
$$(y - 1)\,\{2yS'_{n,m+1}(y) - (2\mu - 1)S_{n,m+1}(y)\} = 2\{ny + \sqrt{y}\mathscr{A}^+\mathscr{B}\}\mathscr{B},$$
$$(y - 1)\,\{y^2 S''_{n,m}(y) + yS'_{n,m}(y) - \mu^2 S_{n,m}(y)\} = 2\sqrt{y}\mathscr{A}^+\mathscr{A}^-\mathscr{B} + m\mathscr{A}^+ + ny\mathscr{A}^-,$$

where

$$\mathscr{A}^{\pm} := yS'_{n,m}(y) \pm \mu S_{n,m}(y), \qquad \mathscr{B} := S_{n+1,m}(y) - S_{n,m+1}(y),$$

and the prime denotes differentiation with respect to y. The analysis of the latter system is similar to that described in the previous subsection regarding system (33) and leads to the following results.

The function $S_{n,m}(y)$ is determined by

$$\frac{\mathrm{d}}{\mathrm{d}y}\ln(S_{n,m}(y)) = \frac{\mu}{y}\frac{y + H_{n,m}(y)}{y - H_{n,m}(y)}, \tag{42i}$$

where the function $H_{n,m}(y)$ is a solution of the equation

$$\mathscr{P}_{\rm VI}\left(y,\, H_{n,m}(y);\, \frac{n^2}{2},\, -\frac{m^2}{2},\, \lambda,\, \frac{1-4\mu^2}{2}\right). \tag{42ii}$$

In the latter, λ stands for a constant of integration which may depend on the parameters n and m.

The functions $S_{n+1,m}(y)$ and $S_{n,m+1}(y)$ are given by

$$S_{n+1,m}(y) = \frac{y^2(y-1)^2H'^2_{n,m} + 2(2\mu-1)y(y-1)H_{n,m}(H_{n,m}-1)H'_{n,m} + A_i H^i_{n,m}}{8\mu(2\mu-1)y^{1/2}(H_{n,m}-y)(H_{n,m}-1)H_{n,m}S_{n,m}}, \tag{43i}$$

$$S_{n,m+1}(y) = \frac{y^2(y-1)^2H'^2_{n,m} + 2(2\mu-1)y^2(y-1)(H_{n,m}-1)H'_{n,m} + B_i H^i_{n,m}}{8\mu(2\mu-1)y^{1/2}(H_{n,m}-y)(H_{n,m}-1)H_{n,m}S_{n,m}}, \tag{43ii}$$

where we have omitted the argument y of $H_{n,m}$ and $S_{n,m}$. In these relations, summation over the repeated index $i = 0, \ldots, 4$ is understood and the coefficients $A_i = A_i(y, n, m)$ and $B_i = B_i(y, n, m)$ are given by

$$\begin{aligned}
&A_0(y,n,m) := -m^2y^2,\\
&A_1(y,n,m) := y[(m^2-2\lambda+(n-2\mu+1)^2)y + 2m^2],\\
&A_2(y,n,m) := -(n-2\mu+1)^2y^2 - 2(m^2-2\lambda+(n-2\mu+1)^2)y + (1-2\mu)^2 - m^2,\\
&A_3(y,n,m) := 2(n-2\mu+1)^2y + m^2 + 2n^2 - 2\lambda - (n+2\mu-1)^2,\\
&A_4(y,n,m) := -n(n-4\mu+2),\\
&B_i(y,n,m) := y^2A_{4-i}(y^{-1},m,n), \qquad i = 0,\ldots,4.
\end{aligned}$$

Remark 4.2. For later purposes, it is noted that the functions $S_{n+1,m}(y)$ and $S_{n,m+1}(y)$ may also be considered as being determined by the Painlevé VI transcendent. Specifically, these functions are determined through the quadratures

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}y}\ln(S_{n+1,m}(y)) &= \frac{1-2\mu}{2y}\,\frac{y+H_{n+1,m}(y)}{y-H_{n+1,m}(y)},\\
\frac{\mathrm{d}}{\mathrm{d}y}\ln(S_{n,m+1}(y)) &= \frac{1-2\mu}{2y}\,\frac{y+H_{n,m+1}(y)}{y-H_{n,m+1}(y)},
\end{aligned} \tag{44i}$$

where $H_{n+1,m}(y)$ and $H_{n,m+1}(y)$ satisfy the equations

$$\mathscr{P}_{\rm VI}\left(y,\, H_{n+1,m}(y);\, \frac{(n+1)^2}{2},\, -\frac{m^2}{2},\, \lambda,\, 2\mu(1-\mu)\right), \tag{44ii}$$

and

$$\mathscr{P}_{\rm VI}\left(y,\, H_{n,m+1}(y);\, \frac{n^2}{2},\, -\frac{(m+1)^2}{2},\, \lambda,\, 2\mu(1-\mu)\right), \tag{44iii}$$

respectively.

This can be proven in the following fashion. Combining relations (43) and (44i), we express $H_{n+1,m}(y)$, $H_{n,m+1}(y)$ in terms of $S_{n,m}(y)$, $G_{n,m}(y)$ and their derivatives. Substituting the resulting expressions into equations (44ii) and (44iii), we arrive at (42ii).

In order to satisfy equation *H1*, the functions u, u_1 and u_2, determined by (41), (42) and (43), respectively, must be such that u_1 and u_2 result from u by applying the shift operators on u, i.e.

$$u_1 = \mathscr{S}_n(u), \qquad u_2 = \mathscr{S}_m(u).$$

These conditions imply that parameter μ must have the form

$$\mu = \tfrac{1}{4}(1 + 2(-1)^{n+m}\tau), \qquad \tau \in \mathbb{R},$$

and functions $S_{i,j}(y)$ must satisfy the following relations:

$$S_{n+1,m}(y) = \mathscr{S}_n(S_{n,m}(y)), \qquad S_{n,m+1}(y) = \mathscr{S}_m(S_{n,m}(y)).$$

It is easily verified that the only consequence of the last conditions is that the parameter λ is independent of n, m. Hence, we conclude that *H1* admits continuously invariant solutions of the form

$$u_{(0,0)} = S_{n,m}\left(\frac{\alpha}{\beta}\right)(\alpha\beta)^{(1+2(-1)^{n+m}\tau)/4}, \qquad \tau \in \mathbb{R}, \tag{45i}$$

where the function $S_{n,m}(y)$ is determined by the quadrature

$$\frac{\mathrm{d}}{\mathrm{d}y}\ln(S_{n,m}(y)) = \frac{1 + 2(-1)^{n+m}\tau}{4y}\,\frac{y + H_{n,m}(y)}{y - H_{n,m}(y)}, \tag{45ii}$$

with $H_{n,m}(y)$ being a solution of

$$\mathscr{P}_{\mathrm{VI}}\left(y, H_{n,m}(y); \frac{n^2}{2}, -\frac{m^2}{2}, \lambda, \frac{1}{2} - \frac{1}{8}(1 + 2(-1)^{n+m}\tau)^2\right), \qquad \lambda \in \mathbb{R}. \tag{45iii}$$

Remark 4.3. Function u, defined in (45), satisfies the invariant surface conditions (38) and, by construction, the differential equation

$$\alpha\partial_\alpha u_{(0,0)} + \beta\partial_\beta u_{(0,0)} = \left(\tfrac{1}{2} + (-1)^{n+m}\tau\right)u_{(0,0)}. \tag{46}$$

Elimination of the derivatives of $u_{(0,0)}$ involved in (38) and (46) leads to

$$\frac{\alpha n}{u_{(1,0)} - u_{(-1,0)}} + \frac{\beta m}{u_{(0,1)} - u_{(0,-1)}} + \left(\frac{1}{2} + (-1)^{n+m}\tau\right)u_{(0,0)} = 0.$$

The last equation implies that solution (45) is also invariant under the action of the generalized symmetry generator $\alpha\mathbf{v}_2 + \beta\mathbf{v}_4 + \tau\mathbf{x}_3$.

Reductions of *H1* using the above symmetry generator were studied in [13], while the connection of the corresponding similarity solutions to discrete generalized and continuous sixth Painlevé equations was demonstrated in [18]. On the other hand, similarity solutions corresponding to the symmetry generated by $\alpha\mathbf{v}_2 + \beta\mathbf{v}_4$ were studied in [22]. The latter are contained in the class of solutions given by (45) for $\tau = 0$.

5. Continuous-invariant solutions of the discrete Schwarzian KdV equation

The $Q1_{\delta=0}$ ABS equation, which is also referred to as *discrete Schwarzian KdV* [21], is given by

$$\alpha(v_{(0,0)} - v_{(0,1)})(v_{(1,0)} - v_{(1,1)}) - \beta(v_{(0,0)} - v_{(1,0)})(v_{(0,1)} - v_{(1,1)}) = 0. \tag{47}$$

In this section, we present continuously invariant solutions of the above equation using similarity solutions of the corresponding system $\Sigma[v]$.

For this reason, we first write out $\Sigma[v]$ explicitly and list the algebra of its Lie point symmetries. Specifically, $\Sigma[v]$ is made up of the equations

$$\frac{\partial v_1}{\partial\beta} = \frac{v_1 - v_2}{\alpha - \beta}\,\frac{m(v - v_1)(v - v_2) - \beta(v_1 - v_2)\frac{\partial v}{\partial\beta}}{(v - v_2)^2}, \tag{48i}$$

$$\frac{\partial v_2}{\partial \alpha} = \frac{v_1 - v_2}{\alpha - \beta} \frac{n(v - v_1)(v - v_2) + \alpha(v_1 - v_2)\frac{\partial v}{\partial \alpha}}{(v - v_1)^2}, \tag{48ii}$$

$$\frac{\partial^2 v}{\partial \alpha \partial \beta} = \frac{1}{\alpha - \beta}\left(2\left(\frac{\alpha}{v - v_1} - \frac{\beta}{v - v_2}\right)\frac{\partial v}{\partial \alpha}\frac{\partial v}{\partial \beta} + n\frac{\partial v}{\partial \beta} - m\frac{\partial v}{\partial \alpha}\right), \tag{48iii}$$

where

$$v = v_{(0,0)}, \qquad v_1 = v_{(1,0)}, \qquad v_2 = v_{(0,1)}.$$

Its algebra of Lie point symmetries is three dimensional and is spanned by the vector fields

$$\mathbf{z}_1 = \alpha\partial_\alpha + \beta\partial_\beta, \qquad \mathbf{z}_2 = \partial_v + \partial_{v_1} + \partial_{v_2}, \qquad \mathbf{z}_3 = v\partial_v + v_1\partial_{v_1} + v_2\partial_{v_2}.$$

The solutions of system (48) remaining invariant under the action of the symmetry generator $\mathbf{z}_1 + 2\gamma\mathbf{z}_3$ are determined by the Painlevé sixth transcendent. To derive this result, we will make use of the following lemma, whose proof is straightforward.

Lemma 5.1. *The contact transformation,*

$$\frac{\partial v}{\partial \alpha} = \frac{u_1 - v}{\alpha}\left(n + (u_1 - v)\frac{\partial u}{\partial \alpha}\right), \tag{49i}$$

$$\frac{\partial v}{\partial \beta} = \frac{u_2 - v}{\beta}\left(m + (u_2 - v)\frac{\partial u}{\partial \beta}\right), \tag{49ii}$$

$$v_1 = u_1, \qquad v_2 = u_2, \tag{49iii}$$

maps solutions of system (28) to solutions of system (48) and conversely.

Thus, we start with the similarity solution of system (28) specified by (41)–(43), in which we set $\lambda = 2\ell^2$, for later convenience. Substitution of this solution to (49) and integration of the result leads to the following solution of $\Sigma[v]$:

$$\begin{aligned} v(\alpha, \beta) &= P_{n,m}(y)z^{-\mu+1/2}, \\ v_1(\alpha, \beta) &= P_{n+1,m}(y)z^{-\mu+1/2}, \\ v_2(\alpha, \beta) &= P_{n,m+1}(y)z^{-\mu+1/2}, \end{aligned} \tag{50}$$

where $y = \alpha/\beta$, $z = \alpha\beta$ and

$$P_{n,m}(y) = \frac{(m + n - 2(\ell + \mu) + 1)(H_{n,m}(y) - y)}{4\mu\sqrt{y}(H_{n,m}(y) - 1)S_{n,m}(y)} - \frac{S_{n+1,m}(y) - H_{n,m}(y)S_{n,m+1}(y)}{H_{n,m}(y) - 1}, \tag{51i}$$

$$P_{n+1,m}(y) = S_{n+1,m}(y), \tag{51ii}$$

$$P_{n,m+1}(y) = S_{n,m+1}(y). \tag{51iii}$$

The functions $P_{i,j}(y)$ are determined by solutions of the sixth Painlevé equation. Regarding $P_{n+1,m}(y)$ and $P_{n,m+1}(y)$, this property is an obvious consequence of equations (51).

On the other hand, $P_{n,m}(y)$ is determined by the solution $\tilde{H}_{n,m}(y)$ of the Painlevé VI equation. More specifically, the former is determined by the latter through the quadrature

$$\frac{\mathrm{d}}{\mathrm{d}y}\ln(P_{n,m}(y)) = \frac{1 - 2\mu}{2y}\frac{y - \tilde{H}_{n,m}(y)}{y + \tilde{H}_{n,m}(y)}, \tag{52i}$$

where $\tilde{H}_{n,m}(y)$ stands for any solution of the Painlevé equation

$$\mathscr{P}_{\rm VI}\left(y, \tilde{H}_{n,m}(y); \frac{n^2}{2}, -\frac{m^2}{2}, \frac{(2\ell-1)^2}{2}, 2\mu(1-\mu)\right). \qquad (52ii)$$

The proof is similar to that described in remark 4.2.

So far, we have shown that the triad of functions

$$v(\alpha,\beta) = P_{n,m}(y)z^{\gamma},$$
$$v_1(\alpha,\beta) = P_{n+1,m}(y)z^{\gamma},$$
$$v_2(\alpha,\beta) = P_{n,m+1}(y)z^{\gamma},$$

where

$$\gamma = -\mu + \tfrac{1}{2},$$

and $P_{i,j}$'s are determined by solutions of the Painlevé VI equation, provides a solution of system (48). Obviously, all members of this triad are also invariant under the symmetry generator $\mathbf{z}_1 + 2\gamma\mathbf{z}_3$, i.e. they satisfy the differential equations

$$\alpha v_{,\alpha} + \beta v_{,\beta} = 2\gamma v, \qquad \alpha v_{1,\alpha} + \beta v_{1,\beta} = 2\gamma v_1, \qquad \alpha v_{2,\alpha} + \beta v_{2,\beta} = 2\gamma v_2,$$

respectively.

The functions v, v_1 and v_2, as defined above, form a solution of the discrete Schwarzian KdV equation provided that

$$v_1 = \mathscr{S}_n(v), \qquad v_2 = \mathscr{S}_m(v).$$

These conditions imply that parameter γ is independent of n, m, and parameter ℓ must have the form

$$\ell = \tfrac{1}{2}(c+n+m+1), \qquad c \in \mathbb{R},$$

in view of which, functions $P_{i,j}(y)$ satisfy the following relations:

$$P_{n+1,m}(y) = \mathscr{S}_n(P_{n,m}(y)), \qquad P_{n,m+1}(y) = \mathscr{S}_m(P_{n,m}(y)).$$

As a result, the continuously invariant solutions of $Q1_{\delta=0}$ constructed above can be written as

$$v_{(0,0)} = P_{n,m}\left(\frac{\alpha}{\beta}\right)(\alpha\beta)^{\gamma}, \qquad \gamma \in \mathbb{R}, \qquad (53i)$$

where

$$\frac{\rm d}{{\rm d}y}\ln(P_{n,m}(y)) = \frac{\gamma}{y}\frac{y + \tilde{H}_{n,m}(y)}{y - \tilde{H}_{n,m}(y)}, \qquad (53ii)$$

and $\tilde{H}_{n,m}(y)$ satisfies the continuous Painlevé VI equation:

$$\mathscr{P}_{\rm VI}\left(y, \tilde{H}_{n,m}(y); \frac{n^2}{2}, -\frac{m^2}{2}, \frac{1}{2}(n+m+c)^2, \frac{1-4\gamma^2}{2}\right), \qquad c \in \mathbb{R}. \quad (53iii)$$

Remark 5.1. The continuously invariant solution (53) also satisfies the differential-difference equations:

$$\alpha\frac{\partial v_{(0,0)}}{\partial\alpha} = n\frac{(v_{(1,0)} - v_{(0,0)})(v_{(0,0)} - v_{(-1,0)})}{v_{(1,0)} - v_{(-1,0)}}, \qquad (54i)$$

$$\beta\frac{\partial v_{(0,0)}}{\partial\beta} = m\frac{(v_{(0,1)} - v_{(0,0)})(v_{(0,0)} - v_{(0,-1)})}{v_{(0,1)} - v_{(0,-1)}}. \qquad (54ii)$$

The latter are but the invariant surface conditions (21*ii*) and (21*iii*). On the other hand, the function $v_{(0,0)}$ also satisfies the differential equation

$$\alpha\partial_\alpha v_{(0,0)} + \beta\partial_\beta v_{(0,0)} = 2\gamma v_{(0,0)}, \tag{55}$$

since it is invariant under the symmetry generator $\mathbf{z}_1 + 2\gamma\mathbf{z}_3$. Using equations (54) to replace the derivatives of $v_{(0,0)}$ appearing in the last equation, we conclude that every continuously invariant solution must satisfy the following constraint:

$$n\frac{(v_{(1,0)} - v_{(0,0)})(v_{(0,0)} - v_{(-1,0)})}{v_{(1,0)} - v_{(-1,0)}} + m\frac{(v_{(0,1)} - v_{(0,0)})(v_{(0,0)} - v_{(0,-1)})}{v_{(0,1)} - v_{(0,-1)}} = 2\gamma v_{(0,0)}.$$

Reductions of the Schwarzian KdV equation constructed on the basis of this constraint were presented in [18].

6. Continuous-invariant solutions and generating equations

The notion of *generating partial differential equations* was introduced by Nijhoff, Joshi and Hone in [19], where their archetypical example, the RPDE, was also presented. In the present section, we show that system $\Sigma[u]$ corresponding to several members of the ABS class is intimately related to the above kind of equations. Our method of deriving this relation enables us to produce the results of [19] in a more systematic way, as well as to extend these results to other integrable lattice equations. In particular, we show that equations *H1–H3* and *Q1* are related to the RPDE.

Remark 6.1. In order to simplify the resulting expressions, in the present section, we adopt the following notation for the corresponding $u_{(i,j)}$:

$$u = u_{(0,0)}, \qquad u_1 = u_{(1,0)}, \qquad u_2 = u_{(0,1)}.$$

6.1. The system $\Sigma[u]$ of H1, H2 and Q1

Let $S(U, u_1, u_2; \delta)$ denote the following system of partial differential equations:

$$\frac{\partial u_1}{\partial\beta} = \frac{u_1 - u_2}{\alpha - \beta}\left(m - (u_1 - u_2)\frac{\partial U}{\partial\beta}\right) + (\alpha - \beta)\delta^2\frac{\partial U}{\partial\beta}, \tag{56i}$$

$$\frac{\partial u_2}{\partial\alpha} = \frac{u_1 - u_2}{\alpha - \beta}\left(n + (u_1 - u_2)\frac{\partial U}{\partial\alpha}\right) - (\alpha - \beta)\delta^2\frac{\partial U}{\partial\alpha}, \tag{56ii}$$

$$\frac{\partial^2 U}{\partial\alpha\partial\beta} = \frac{1}{\alpha - \beta}\left(2(u_1 - u_2)\frac{\partial U}{\partial\alpha}\frac{\partial U}{\partial\beta} + n\frac{\partial U}{\partial\beta} - m\frac{\partial U}{\partial\alpha}\right). \tag{56iii}$$

One arrives at the above system starting from $\Sigma[u]$ corresponding to equations *H1*, *H2* and *Q1* through the following contact transformation:

$$\mathscr{M}(u, U) := \begin{cases} u_{,\alpha} = \dfrac{2f(u, u_1, \alpha)U_{,\alpha} + nf_{,u_1}(u, u_1, \alpha)}{2r(\alpha)} \\ u_{,\beta} = \dfrac{2f(u, u_2, \beta)U_{,\beta} + mf_{,u_2}(u, u_2, \beta)}{2r(\beta)}. \end{cases} \tag{57}$$

In particular, $\mathscr{M}(u, U)$ maps

(1) $\Sigma[u]$ of *H1* to $S[u; 0]$,
(2) $\Sigma[u]$ of *H2* to $S[U; \delta]$ with $\delta^2 = 1$, and
(3) $\Sigma[u]$ of *Q1* to $S[U; \delta]$.

In view of these observations, system $S[U;\delta]$ introduced above incorporates the continuously invariant solutions of the three integrable lattice equations *H1*, *H2* and *Q1*. What is remarkable is the fact that $S[U;\delta]$ is also related to the RPDE, which has been shown to be a generating equation for the KdV hierarchy [19].

Indeed, $S[U;\delta]$ can be decoupled leading to a fourth-order partial differential equation for each of the functions involved. To see this, we first solve equation (56*iii*) for the difference $u_1 - u_2$ to find

$$u_1 - u_2 = \frac{1}{2}\left((\alpha-\beta)\frac{U_{,\alpha\beta}}{U_{,\alpha}U_{,\beta}} + \frac{m}{U_{,\beta}} - \frac{n}{U_{,\alpha}}\right). \tag{58}$$

Substituting the above expression into (56*i*) and (56*ii*), we obtain $\partial_\beta u_1$ and $\partial_\alpha u_2$ in terms of U and its derivatives. Then, we differentiate equation (58) with respect to α and use (56*ii*) and (58) to eliminate $\partial_\alpha u_2$ and $u_1 - u_2$, respectively. This gives $\partial_\alpha u_1$ in terms of the derivatives of U. The compatibility between the resulting expression and the first equation of $S[U;\delta]$ leads to the following fourth-order partial differential equation:

$$\mathscr{R}(\alpha,\beta,U;n,m) + 2\delta^2\frac{\partial U}{\partial\alpha}\frac{\partial U}{\partial\beta}\left(2\frac{\partial^2 U}{\partial\alpha\partial\beta} - \frac{1}{\alpha-\beta}\left(\frac{\partial U}{\partial\alpha} - \frac{\partial U}{\partial\beta}\right)\right) = 0, \tag{59}$$

where

$$\begin{aligned}
\mathscr{R}(\alpha,\beta,U;n,m) :=& -U_{,\alpha\alpha\beta\beta} + U_{,\alpha\alpha\beta}\left(\frac{1}{\alpha-\beta} + \frac{U_{,\beta\beta}}{U_{,\beta}} + \frac{U_{,\alpha\beta}}{U_{,\alpha}}\right)\\
&+ U_{,\alpha\beta\beta}\left(\frac{1}{\beta-\alpha} + \frac{U_{,\alpha\alpha}}{U_{,\alpha}} + \frac{U_{,\alpha\beta}}{U_{,\beta}}\right) - U_{,\alpha\alpha}U_{,\beta\beta}\frac{U_{,\alpha\beta}}{U_{,\alpha}U_{,\beta}}\\
&+ U_{,\alpha\alpha}\left(\frac{n^2}{(\alpha-\beta)^2}\frac{U_{,\beta}^2}{U_{,\alpha}^2} - \frac{1}{\alpha-\beta}\frac{U_{,\alpha\beta}}{U_{,\alpha}} - \frac{U_{,\alpha\beta}^2}{U_{,\alpha}^2}\right)\\
&+ U_{,\beta\beta}\left(\frac{m^2}{(\alpha-\beta)^2}\frac{U_{,\alpha}^2}{U_{,\beta}^2} + \frac{1}{\alpha-\beta}\frac{U_{,\alpha\beta}}{U_{,\beta}} - \frac{U_{,\alpha\beta}^2}{U_{,\beta}^2}\right)\\
&+ \frac{n^2}{2(\alpha-\beta)^3}\frac{U_{,\beta}}{U_{,\alpha}}(U_{,\alpha} + U_{,\beta} + 2(\beta-\alpha)U_{,\alpha\beta})\\
&- \frac{m^2}{2(\alpha-\beta)^3}\frac{U_{,\alpha}}{U_{,\beta}}(U_{,\alpha} + U_{,\beta} + 2(\alpha-\beta)U_{,\alpha\beta}) + \frac{1}{2(\alpha-\beta)}U_{,\alpha\beta}^2\left(\frac{1}{U_{,\alpha}} - \frac{1}{U_{,\beta}}\right).
\end{aligned}$$

When $\delta = 0$, the last equation reduces to $\mathscr{R}(\alpha,\beta,U;n,m) = 0$, and this is exactly the equation named the RPDE [19, 23]. In fact, even when $\delta \neq 0$, equation (59) is essentially the same to the RPDE. Specifically, one only needs to set

$$\tilde{U} = \frac{1}{2\delta}\exp(2\delta U),$$

in order to transform equation (59) to $\mathscr{R}(\alpha,\beta,\tilde{U};n,m) = 0$. For this reason, equation (59) will be referred to as the RPDE for all values of parameter δ.

The function U may also be considered as a potential for u_1 and u_2. To see this, we just have to solve $S[U;\delta]$ for the derivatives of U. The compatibility of the resulting equations leads to the following system for u_1, u_2:

$$\begin{aligned}
u_{1,\alpha\beta} =& \frac{2(u_1-u_2)}{(u_1-u_2)^2 - \delta^2(\alpha-\beta)^2}(u_{1,\alpha}u_{1,\beta} + 2\delta^2(n+1)m)\\
&- \left(\frac{2\delta^2(\alpha-\beta)}{(u_1-u_2)^2 - \delta^2(\alpha-\beta)^2} + \frac{1}{\alpha-\beta}\right)(mu_{1,\alpha} + (n+1)u_{1,\beta}),
\end{aligned} \tag{60i}$$

$$u_{2,\alpha\beta} = \frac{2(u_2-u_1)}{(u_1-u_2)^2-\delta^2(\alpha-\beta)^2}(u_{2,\alpha}u_{2,\beta}+2\delta^2 n(m+1)) + \left(\frac{2\delta^2(\alpha-\beta)}{(u_1-u_2)^2-\delta^2(\alpha-\beta)^2}+\frac{1}{\alpha-\beta}\right)((m+1)u_{2,\alpha}+nu_{2,\beta}). \qquad (60ii)$$

When $\delta = 0$, the last equations decouple easily yielding the RPDE pair:

$$\mathscr{R}(\alpha,\beta,u_1;n+1,m)=0, \qquad \mathscr{R}(\alpha,\beta,u_2;n,m+1)=0.$$

In the case $\delta \neq 0$, system (60) may also be decoupled but the resulting equations are much more complicated. Specifically, one may solve equation ($60i$) for u_2 and substitute the result into equation ($60ii$). This leads to a fourth-order, second-degree partial differential equation for u_1, which is omitted here because of its length. Analogous considerations hold for the function u_2.

Remark 6.2. System $S[U;0]$ first appeared in [19]. A generalization of $S[U;0]$ was derived in [23, 24] in the context of a symmetry reduction of the anti-self-dual Yang–Mills equations. The relation of the latter to the Ernst–Weyl equation and the Painlevé transcendents was also presented in [24].

Remark 6.3. As already noted, $S[U;\delta]$ is integrable. A Lax pair for this system is given by

$$\Psi_{,\alpha} = \frac{1}{\alpha-\lambda}\begin{pmatrix} n+u_1U_{,\alpha} & -U_{,\alpha} \\ u_1(n+u_1U_{,\alpha}) & -u_1U_{,\alpha}\end{pmatrix}\Psi - \begin{pmatrix} 0 & 0 \\ \delta^2(\alpha-\lambda)U_{,\alpha} & 0\end{pmatrix}\Psi, \qquad (61i)$$

$$\Psi_{,\beta} = \frac{1}{\beta-\lambda}\begin{pmatrix} m+u_2U_{,\beta} & -U_{,\beta} \\ u_2(m+u_2U_{,\beta}) & -u_2U_{,\beta}\end{pmatrix}\Psi - \begin{pmatrix} 0 & 0 \\ \delta^2(\beta-\lambda)U_{,\beta} & 0\end{pmatrix}\Psi. \qquad (61ii)$$

It can be obtained from the Lax pair (26) using the transformation $\mathscr{M}$ and performing the gauge transformation

$$\Phi = (\alpha-\lambda)^{-n/2}(\beta-\lambda)^{-m/2}\Psi.$$

Remark 6.4. It is worth mentioning that equation (59) is the Euler–Lagrange equations:

$$\frac{\partial^2}{\partial\alpha\partial\beta}\left(\frac{\partial\mathscr{L}}{\partial U_{,\alpha\beta}}\right) - \frac{\partial}{\partial\alpha}\left(\frac{\partial\mathscr{L}}{\partial U_{,\alpha}}\right) - \frac{\partial}{\partial\beta}\left(\frac{\partial\mathscr{L}}{\partial U_{,\beta}}\right) = 0$$

corresponding to the Lagrangian

$$\mathscr{L} = \frac{\alpha-\beta}{2}\frac{U_{,\alpha\beta}^2}{U_{,\alpha}U_{,\beta}} + \frac{1}{2(\alpha-\beta)}\left(m^2\frac{U_{,\alpha}}{U_{,\beta}}+n^2\frac{U_{,\beta}}{U_{,\alpha}}\right) + 2\delta^2(\alpha-\beta)U_{,\alpha}U_{,\beta}.$$

For $\delta = 0$, this reduces to the Lagrangian for the RPDE given in [19].

6.2. The system $\Sigma[u]$ of H3

The continuously invariant solutions of *H3* are also related to solutions of the RPDE. We establish this connection for the cases $\delta = 0$ and $\delta \neq 0$ separately. In each case, we introduce a potential function through a system of equations and use the latter to simplify the corresponding system $\Sigma[u]$. The resulting system can be decoupled leading to the RPDE for the potential function.

6.2.1. Case I: $\delta = 0$. Let us first introduce a potential ψ for system $\Sigma[u]$ corresponding to $H3_{\delta=0}$. This is determined by the relations

$$\psi_{,\alpha} = \frac{e^U(n - \alpha U_{,\alpha})}{2u_1}, \qquad \psi_{,\beta} = \frac{e^U(m - \beta U_{,\beta})}{2u_2}, \tag{62}$$

where

$$\exp(-U(\alpha, \beta)) = u(\alpha, \beta).$$

We solve the above equations for u_1, u_2 and substitute the resulting expressions into the first two equations of $\Sigma[u]$. Then, we perform the change of the independent variables

$$(\alpha, \beta) \longrightarrow (\alpha^2, \beta^2)$$

and arrive at the following system for U and ψ:

$$U_{,\alpha\beta} = \frac{1}{4(\alpha-\beta)}\left(\frac{4\alpha^2U_{,\alpha}^2 - n^2}{\alpha}\frac{\psi_{,\beta}}{\psi_{,\alpha}} - \frac{4\beta^2U_{,\beta}^2 - m^2}{\beta}\frac{\psi_{,\alpha}}{\psi_{,\beta}}\right), \tag{63i}$$

$$\psi_{,\alpha\beta} = \frac{2}{\alpha-\beta}(\alpha U_{,\alpha}\psi_{,\beta} - \beta U_{,\beta}\psi_{,\alpha}). \tag{63ii}$$

On the other hand, using the above substitutions for u, u_1 and u_2 and the change of the independent variables, the third equation of $\Sigma[u]$ is identically satisfied by taking into account system (63).

The pair of equations (63) can be decoupled, and this leads to the following equation for the potential ψ:

$$\mathscr{R}(\alpha, \beta, \psi; n, m) = 0.$$

The decoupling can be achieved by solving equation (63ii) for one of the first-order derivatives of U, e.g. $U_{,\alpha}$, and taking the compatibility condition between the resulting equation and (63i). The result is a relation for $U_{,\beta\beta}$. Finally, the compatibility condition between the latter and (63i) implies that ψ satisfies the RPDE.

On the other hand, system (63) may be decoupled leading to a fourth-order, second-degree partial differential equation for function U. First, we solve (63i) for $\psi_{,\alpha}$ to get

$$\psi_{,\alpha} = \psi_{,\beta}A, \qquad \psi_{,\alpha\beta} = \psi_{,\beta}B,$$

where

$$A = \frac{2\alpha\beta(\alpha-\beta)U_{,\alpha\beta} + X}{\alpha(m^2 - \beta^2U_{,\beta}^2)}, \qquad B = \frac{2(\alpha U_{,\alpha} - \beta A U_{,\beta})}{\alpha - \beta}$$

and

$$X = \sqrt{\alpha\beta\left(4\alpha\beta(\alpha-\beta)^2U_{,\alpha\beta}^2 + \left(m^2 - \beta^2U_{,\beta}^2\right)\left(n^2 - \alpha^2U_{,\alpha}^2\right)\right)}.$$

The compatibility condition $\partial_\beta\psi_{,\alpha} = \psi_{,\alpha\beta}$ implies

$$\psi_{,\beta\beta} = \psi_{,\beta}\left(\frac{B - D_\beta A}{A}\right).$$

Finally, the compatibility condition $\partial_\beta\psi_{,\alpha\beta} = \partial_\alpha\psi_{,\beta\beta}$ leads to

$$D_\alpha D_\beta \ln A = D_\alpha\left(\frac{B}{A}\right) - D_\beta B.$$

If we write out the last equation explicitly, solve it for X and square the result, then we end up with a fourth-order, second-degree (in the highest derivative $U_{,\alpha\alpha\beta\beta}$) partial differential equation. It is the *modified partial differential equation* (MPDE) presented in [19].

6.2.2. Case II: $\delta \neq 0$. In this case, we introduce the potential ϕ by the relations

$$\phi_{,\alpha} = \frac{u_1(nu + \alpha u_{,\alpha})}{\alpha(uu_1 + \delta\alpha)}, \qquad \phi_{,\beta} = \frac{u_2(mu + \beta u_{,\beta})}{\beta(uu_2 + \delta\beta)},$$

in view of which u_1, u_2 may be expressed in terms of u, ϕ and their derivatives:

$$u_1 = \frac{\delta\alpha^2\phi_{,\alpha}}{\alpha u_{,\alpha} + u(n - \alpha\phi_{,\alpha})}, \qquad u_2 = \frac{\delta\beta^2\phi_{,\beta}}{\beta u_{,\beta} + u(m - \beta\phi_{,\beta})}. \tag{64}$$

We substitute the above relations into the first two equations of $\Sigma[u]$ and set

$$\begin{aligned} u(\alpha, \beta) &:= \exp(-U(\alpha, \beta) + \delta\psi(\alpha, \beta)), \\ \phi(\alpha, \beta) &:= -U(\alpha, \beta) - \delta\psi(\alpha, \beta) + n\ln\alpha + m\ln\beta. \end{aligned} \tag{65}$$

Finally, the change of the independent variables

$$(\alpha, \beta) \longrightarrow (\alpha^2, \beta^2) \tag{66}$$

leads to the following system:

$$U_{,\alpha\beta} = \frac{1}{4(\alpha - \beta)}\left(\frac{4\alpha^2U_{,\alpha}^2 - n^2}{\alpha}\frac{\psi_{,\beta}}{\psi_{,\alpha}} - \frac{4\beta^2U_{,\beta}^2 - m^2}{\beta}\frac{\psi_{,\alpha}}{\psi_{,\beta}}\right) + \delta^2\psi_{,\alpha}\psi_{,\beta}, \tag{67i}$$

$$\psi_{,\alpha\beta} = \frac{2}{\alpha - \beta}(\alpha U_{,\alpha}\psi_{,\beta} - \beta U_{,\beta}\psi_{,\alpha}). \tag{67ii}$$

Moreover, the third equation of $\Sigma[u]$ is satisfied identically, in view of transformations (64)–(66) and by taking into account system (67).

We may decouple system (67) following the procedure described in the previous subsection. In this fashion, we conclude that ψ satisfies the RPDE, as well.

Remark 6.5. A Lax pair for system (67) is given by

$$\begin{aligned} \Psi_{,\alpha} &= \frac{1}{\alpha - \lambda^2}\begin{pmatrix} \alpha U_{,\alpha} + \lambda^2\delta\psi_{,\alpha} & 2\lambda\psi_{,\alpha} \\ \frac{\lambda}{8\alpha\psi_{,\alpha}}(n^2 - 4\alpha^2(U_{,\alpha} + \delta\psi_{,\alpha})^2) & -\alpha U_{,\alpha} - \lambda^2\delta\psi_{,\alpha} \end{pmatrix}\Psi, \\ \Psi_{,\beta} &= \frac{1}{\beta - \lambda^2}\begin{pmatrix} \beta U_{,\beta} + \lambda^2\delta\psi_{,\beta} & 2\lambda\psi_{,\beta} \\ \frac{\lambda}{8\beta\psi_{,\beta}}(m^2 - 4\beta^2(U_{,\beta} + \delta\psi_{,\beta})^2) & -\beta U_{,\beta} - \lambda^2\delta\psi_{,\beta} \end{pmatrix}\Psi. \end{aligned} \tag{68}$$

The above equations follow from (26) by making the transformations (64)–(66) and, subsequently, performing the gauge transformation:

$$\Phi = \begin{pmatrix} \exp\left(\frac{\delta\psi - U}{2}\right) & 0 \\ 0 & \exp\left(\frac{U - \delta\psi}{2}\right) \end{pmatrix}\Psi.$$

6.3. Connection with previous results

The preceding analysis shows that the integrable lattice equations *H1–H3* and *Q1* are closely related, i.e. their continuously invariant solutions may be expressed in terms of solutions of the RPDE.

The relation among *H1*, $H3_{\delta=0}$ and $Q1_{\delta=0}$ and the RPDE was presented by Nijhoff, Hone and Joshi in [19], starting from a different point of view. Specifically, the authors presented systems of differential–difference equations compatible with the above lattice equations, which, from our point of view, are the invariant surface conditions (21ii) and (21iii). Using the differential–difference equations, they constructed compatible systems of partial differential equations, which in turn lead to the RPDE and MPDE. Actually, the systems appearing in the analysis of Nijhoff, Hone and Joshi do not differ from what we called $\Sigma[u]$.

To clarify this correspondence further, let us point out the following.

(i) The system presented in [19] in relation with *H1* is actually $S(u, -u_1, -u_2; 0)$.

(ii) In relation with $Q1_{\delta=0}$, the authors of [19] presented a system of differential equations Δ, which can be decoupled leading to a fourth-order partial differential equation, called Schwarzian partial differential equations (SPDEs). Here, we have presented the corresponding system $\Sigma[u]$, i.e. system (48), which may be decoupled for each involved function leading to the RPDE.

However, system Δ and the resulting SPDE are related to (48) and the RPDE, respectively. Indeed, starting from system (48), we make the change of the dependent variables

$$u_1 = u + \frac{2\alpha u_{,\alpha}}{n(1-\tilde{u}_1)}, \qquad u_2 = u + \frac{2\beta u_{,\beta}}{m(1-\tilde{u}_2)},$$

and, consequently, the change of the independent variables

$$(\alpha, \beta) \longrightarrow \left(\frac{1}{\alpha}, \frac{1}{\beta}\right).$$

This procedure leads to system Δ. Moreover, the RPDE is mapped to the SPDE using the above transformation of the independent variables α, β.

(iii) Finally, the authors of [19] also presented the MPDE in relation with $H3_{\delta=0}$ and a Miura transformation relating the MPDE to the RPDE. From our point of view, this Miura transformation is system (63), which is equivalent to $\Sigma[u]$ corresponding to $H3_{\delta=0}$.

7. Conclusions and perspectives

We have presented symmetry reductions of the Adler, Bobenko and Suris equations using both of the extended three-point generalized symmetries admitted by them. Such reductions lead to special similarity solutions, which we named continuously invariant solutions. It was proven that these are determined by a system of partial differential equations, $\Sigma[u]$, which is integrable in the sense that it admits an auto-Bäcklund transformation and a Lax pair.

The symmetry analysis and the corresponding reductions of system $\Sigma[u]$ associated with the discrete potential and Schwarzian KdV equations led to new interesting results. In particular, it was shown that the continuously invariant solutions of *H1* are determined by solutions of the continuous Painlevé V and VI equations. Similar results and considerations were also presented with regard to equation $Q1_{\delta=0}$.

We were also able to reveal the connection of $\Sigma[u]$ to generating equations. In particular, we derived the generating equations to the *H1*–*H3* and *Q1* members of the ABS family. In addition, we showed that the continuously invariant solutions of *H1*, $H3_{\delta=0}$ and $Q1_{\delta=0}$ are related to the RPDE, in accordance with the results of Nijhoff, Hone and Joshi in [19].

The construction of continuously invariant solutions of the other members of the ABS family and, especially, of the master equation *Q4* is one of the interesting directions in which the present work can be extended. In addition, more general lattice systems possessing the consistency property can also be analyzed in the framework of continuously invariant solutions and generating equations. The discrete Boussinesq equation [25] and the discrete modified Boussinesq equation [26] are among the better known systems which can be brought into the above framework.

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Appendix A. The characteristic polynomials of the ABS equations

H1: $f(u,x,\alpha)=1 \quad k(\alpha,\beta)=\beta-\alpha \quad G(x,y)=(x-y)^2$

H2: $f(u,x,\alpha)=2(u+x+\alpha) \quad k(\alpha,\beta)=\beta-\alpha \quad G(x,y)=(x-y)^2-(\alpha-\beta)^2$

H3: $f(u,x,\alpha)=ux+\alpha\delta \quad k(\alpha,\beta)=\alpha^2-\beta^2 \quad G(x,y)=(y\alpha-x\beta)(y\beta-x\alpha)$

Q1: $f(u,x,\alpha)=((u-x)^2-\alpha^2\delta^2)/\alpha, \qquad k(\alpha,\beta)=-\alpha\beta(\alpha-\beta)$

$G(x,y)=\alpha\beta((x-y)^2-(\alpha-\beta)^2\delta^2)$

Q2: $f(u,x,\alpha)=((u-x)^2-2\alpha^2(u+x)+\alpha^4)/\alpha, \qquad k(\alpha,\beta)=-\alpha\beta(\alpha-\beta)$

$G(x,y)=\alpha\beta((x-y)^2-2(\alpha-\beta)^2(x+y)+(\alpha-\beta)^4)$

Q3: $f(u,x,\alpha)=\dfrac{1}{4\alpha(\alpha^2-1)}(4\alpha(\alpha u-x)(\alpha x-u)-(\alpha^2-1)^2\delta^2)$

$k(\alpha,\beta)=(\alpha^2-\beta^2)(\alpha^2-1)(\beta^2-1)$

$G(x,y)=\dfrac{(\alpha^2-1)(\beta^2-1)}{4\alpha\beta}(4\alpha\beta(\alpha y-\beta x)(\beta y-\alpha x)+(\alpha^2-\beta^2)\delta^2)$

Q4: $f(u,x,\alpha)=((ux+\alpha(u+x)+g_2/4)^2-(u+x+\alpha)(4\alpha ux-g_3))/a$

$k(\alpha,\beta)=\dfrac{ab(a^2b+ab^2+[12\alpha\beta^2-g_2(\alpha+2\beta)-3g_3]a+[12\beta\alpha^2-g_2(\beta+2\alpha)-3g_3]b)}{4(\alpha-\beta)}$

$$G(x,y)=(a_0xy+a_1(x+y)+a_2)(a_2xy+a_3(x+y)+a_4) \\ -(a_1xy+\bar{a}_2y+\tilde{a}_2x+a_3)(a_1xy+\bar{a}_2x+\tilde{a}_2y+a_3)$$

Appendix B. Proof of proposition 3.2

Let u be a continuously invariant solution of the integrable lattice equation

$$Q(u_{(0,0)},u_{(1,0)},u_{(0,1)},u_{(1,1)};\alpha,\beta)=0 \tag{B.1}$$

and $\tilde{u}$ be constructed in terms of u via the auto-Bäcklund transformation $\mathbb{B}_d(u,\tilde{u},\lambda)$. The function $\tilde{u}$ is another continuously invariant solution of equation (B.1), provided that it satisfies the system

$$Q(\tilde{u}_{(0,0)},\tilde{u}_{(1,0)},\tilde{u}_{(0,1)},\tilde{u}_{(1,1)};\alpha,\beta)=0, \tag{B.2i}$$

$$r(\alpha)\frac{\partial\tilde{u}_{(0,0)}}{\partial\alpha}+nR(\tilde{u}_{(0,0)},\tilde{u}_{(1,0)},\tilde{u}_{(-1,0)},\alpha)=0, \tag{B.2ii}$$

$$r(\beta)\frac{\partial\tilde{u}_{(0,0)}}{\partial\beta}+mR(\tilde{u}_{(0,0)},\tilde{u}_{(0,1)},\tilde{u}_{(0,-1)},\beta)=0. \tag{B.2iii}$$

Obviously, the first equation holds, since $\tilde{u}$ is constructed using $\mathbb{B}_d(u,\tilde{u},\lambda)$. It remains to show that equations (B.2ii) and (B.2iii) also hold.

To prove that (B.2ii) holds, we differentiate the first equation of $\mathbb{B}_d(u, \tilde{u}, \lambda)$ with respect to α. Then, we use equation (21ii) and its shift in the n direction to substitute $\partial_\alpha u_{(0,0)}$ and $\partial_\alpha u_{(1,0)}$, respectively. Moreover, we use the determining equation for the generator $\tilde{\mathbf{v}}_1$ to substitute the derivative of $Q(u_{(0,0)}, u_{(1,0)}, \tilde{u}_{(0,0)}, \tilde{u}_{(1,0)}; \alpha, \lambda)$ with respect to α. In terms of the above substitutions, we come up with the following equation:

$$\left(Q_{,\tilde{u}_{(0,0)}} + Q_{,\tilde{u}_{(1,0)}}\mathcal{S}_n\right)\left(r(\alpha)\frac{\partial \tilde{u}_{(0,0)}}{\partial\alpha} + nR(\tilde{u}_{(0,0)}, \tilde{u}_{(1,0)}, \tilde{u}_{(-1,0)}, \alpha)\right) = 0,$$

where we have omitted the arguments of the function $Q(u_{(0,0)}, u_{(1,0)}, \tilde{u}_{(0,0)}, \tilde{u}_{(1,0)}; \alpha, \lambda)$.

Finally, we eliminate the value $u_{(1,0)}$ from the above equation and the latter becomes

$$(G(u_{(0,0)}, \tilde{u}_{(1,0)}) + h(u_{(0,0)}, \tilde{u}_{(0,0)})\mathcal{S}_n)\left(r(\alpha)\frac{\partial \tilde{u}_{(0,0)}}{\partial\alpha} + nR(\tilde{u}_{(0,0)}, \tilde{u}_{(1,0)}, \tilde{u}_{(-1,0)}, \alpha)\right) = 0.$$

This equation involves the values of the function $\tilde{u}$ and the value $u_{(0,0)}$ through the polynomials h, G. Thus, the corresponding coefficients of the various powers of $u_{(0,0)}$ must be identically zero. The matrix of the resulting algebraic system has rank 2 [13], and the system admits only the zero solution, i.e.

$$r(\alpha)\frac{\partial \tilde{u}_{(0,0)}}{\partial\alpha} + nR(\tilde{u}_{(0,0)}, \tilde{u}_{(1,0)}, \tilde{u}_{(-1,0)}, \alpha) = 0. \tag{B.3i}$$

In the same fashion, we differentiate the second equation of the auto-Bäcklund transformation with respect to β and use equation (21iii) and the determining equation for the symmetry generator $\tilde{\mathbf{v}}_2$ to get that $\tilde{u}$ also satisfies

$$r(\beta)\frac{\partial \tilde{u}_{(0,0)}}{\partial\beta} + mR(\tilde{u}_{(0,0)}, \tilde{u}_{(0,1)}, \tilde{u}_{(0,-1)}, \beta) = 0. \tag{B.3ii}$$

Thus, function $\tilde{u}$ satisfies system (B.2), i.e. it is another continuously invariant solution.

Appendix C. Proof of proposition 3.3

Proposition 3.2 implies that if u is a continuously invariant solution, then function $\tilde{u}$, determined by $\mathbb{B}_d(u, \tilde{u}, \lambda)$, will be another solution of the same kind, and conversely. In other words, the functions u and $\tilde{u}$ satisfy systems (21) and (B.2), respectively. Thus, we can express the derivatives of $\tilde{u}_{(0,0)}$ in terms of the values and the corresponding derivatives of the function u and conversely.

To achieve this, we solve

$$Q(u_{(-1,0)}, u_{(0,0)}, \tilde{u}_{(-1,0)}, \tilde{u}_{(0,0)}; \alpha, \lambda) = 0 \tag{C.1}$$

for $u_{(-1,0)}$ and (B.2ii) for $\tilde{u}_{(-1,0)}$. In terms of the above substitutions, the fraction $1/(u_{(1,0)} - u_{(-1,0)})$ becomes

$$\left(\frac{r(\alpha)}{n}\partial_\alpha \tilde{u}_{(0,0)} - \frac{f_{,\tilde{u}_{(1,0)}}(\tilde{u}_{(0,0)}, \tilde{u}_{(1,0)}, \alpha)}{2}\right)\frac{Q_{,u_{(1,0)}}}{Q_{,\tilde{u}_{(1,0)}} f(\tilde{u}_{(0,0)}, \tilde{u}_{(1,0)}, \alpha)} + \frac{Q_{,u_{(1,0)}\tilde{u}_{(1,0)}}}{Q_{,\tilde{u}_{(1,0)}}},$$

where we have omitted the arguments of $Q(u_{(0,0)}, u_{(1,0)}, \tilde{u}_{(0,0)}, \tilde{u}_{(1,0)}; \alpha, \lambda)$.

The derivatives of Q involved in the above expression are determined by the relations

$$\frac{Q_{,u_{(1,0)}}}{Q_{,\tilde{u}_{(1,0)}}} = \frac{k(\alpha,\lambda) f(\tilde{u}_{(0,0)},\tilde{u}_{(1,0)},\alpha)}{G(u_{(1,0)},\tilde{u}_{(0,0)},\alpha,\lambda)},$$

$$\frac{Q_{,u_{(1,0)}\tilde{u}_{(1,0)}}}{Q_{,\tilde{u}_{(1,0)}}} = \frac{1}{2}\left(\frac{k(\alpha,\lambda) f_{,\tilde{u}_{(1,0)}}(\tilde{u}_{(0,0)},\tilde{u}_{(1,0)},\alpha)}{G(u_{(1,0)},\tilde{u}_{(0,0)},\alpha,\lambda)} + \frac{G_{,u_{(1,0)}}(u_{(1,0)},\tilde{u}_{(0,0)},\alpha,\lambda)}{G(u_{(1,0)},\tilde{u}_{(0,0)},\alpha,\lambda)}\right),$$

which hold in view of the equation $Q(u_{(0,0)},u_{(1,0)},\tilde{u}_{(0,0)},\tilde{u}_{(1,0)};\alpha,\lambda)=0$.

We arrive at equation ($25iii$) by substituting the above expressions into ($21ii$) and solving the resulting equation for $\partial_\alpha \tilde{u}_{(0,0)}$.

Conversely, we solve (C.1) for $\tilde{u}_{(-1,0)}$ and ($21ii$) for $u_{(-1,0)}$. Then, we substitute the resulting expressions into (B.$2ii$) and solve this equation for $\partial_\alpha u_{(0,0)}$. The final result is identical to the equation obtained by interchanging u and $\tilde{u}$ in ($25iii$).

Equation ($25iv$), i.e. the fourth equation of $\mathbb{B}_c(u,\tilde{u},\lambda)$, can be derived in a similar manner using equations $Q(u_{(0,-1)},u_{(0,0)},\tilde{u}_{(0,-1)},\tilde{u}_{(0,0)};\beta,\lambda)=0$ and (B.$2iii$).

Since the class of continuously invariant solutions is closed under $\mathbb{B}_d$ and the initial solution u satisfies $\Sigma[u]$, the same holds for the function $\tilde{u}$, i.e. the latter satisfies $\Sigma[\tilde{u}]$. Thus, $\mathbb{B}_c(u,\tilde{u},\lambda)$ defines an auto-Bäcklund transformation of system $\Sigma[u]$.

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